Physics II Notes

TABLE OF CONTENTS

1 ELECTROSTATICS PART 1

Charge is a quantized property, incrementing in orders of $q_p = -q_e = 1.602 \times 10^{-19} C$. Net charge of matter is EXACTLY zero and charge is always conserved in a closed system.

Charge does not depend on relativistic speed.

“Electrostatic” means that all charges are stationary in the observed reference frame.

$$\vec{F}_{1\to 2} = -\vec{F}_{2\to 1} = k\frac{Q_1 Q_2}{r^2}\vec{u}_{r_1}$$
$$k = \frac{1}{4\pi\varepsilon_0} = 8.9875 \times 10^9 Nm^2C^{-2}$$
$$\varepsilon_0 = \text{Vacuum Electrical Permittivity} = 8.85 \times 10^{-12} C^2 m^{-1} m^2$$

An electrostatic force is a central, conservative force. A vector field (electrostatic field) can be defined, for which Gauss Law will hold, but it is not an acceleration field.

A scalar potential (Electrostatic potential) can be defined.

Electrostatic force can be either repulsive or attractive.

Electrostatic field with unit Vm^{-1} is defined as:

$$\vec{E} = \frac{kQ}{r^2}\vec{u}_r$$
$$\Rightarrow F_{es} = q\vec{E}$$

The electrostatic field:

- Is radial
- Is measured in NC^{-1}
- To measure it, a sample charge is needed, on which the force is measured.
- Has an orientation, that depends on the charge sign. Outgoing if the charge is positive, ingoing if the charge is negative.
- Is not an acceleration field because acceleration depends on mass, and electrostatic force is unaffected by mass.

Electrostatic force is conservative (Word done along a closed path is 0), so electrostatic potential energy can be defined:

$$E_{POT} = U = k\frac{Qq}{r} + \text{constant}$$

If charges have the same sign, energy is positive and is minimized when charges are at infinite distance. If charges are of opposite sign, energy is negative and is minimized when the charges are in contact.

An electrostatic potential can be defined as well, with unit being the volt V:

$$V_{es} = k\frac{Q}{r}$$

$$\frac{\vec{F}}{q} = \vec{E} \qquad \vec{F} = -\nabla U \qquad U = -\int \vec{F} \cdot d\vec{s}$$

$$V_{es} = \frac{U}{q} \qquad \vec{E} = -\nabla V_{es} \qquad V_{es} = -\int \vec{E} \cdot d\vec{s}$$

Electrostatic potential difference between two points A and B is evaluated as:

$$\Delta V_{AB} = -\int_A^B \vec{E}\, d\vec{s} = V_A - V_B$$

If the path is closed ($\vec{A} = \vec{B}$), then $\Delta V_{AB} = 0$

Circulation of an electrostatic field is zero:

$$\oint \vec{E}\, d\vec{s} = 0$$

This is fully equivalent to “Electrostatic force is conservative”

For a system of point charges:

$$E_{TOT} = \sum \vec{E}_i, \qquad V_{TOT} = \sum v_i$$
$$\vec{F}_{TOT} = Q_P \vec{E}_{TOT}, \qquad U_{TOT} = Q_P V_{TOT}$$

For a body with charge density ρ (points have position $\vec{r}$ relative to an origin O). To evaluate potential at a displacement of $\vec{r}'$ from the origin, we write:

$$\vec{E}_{TOT}(\vec{r}') = -\nabla_{\vec{r}'} \cdot V_{TOT}(\vec{r}') = \int_V \left[k \frac{\rho(\vec{r}_O)}{|\vec{r}' - \vec{r}_O|^2} \vec{u}_r \right] dV, \qquad V_{TOT}(\vec{r}') = \int_V \left[k \frac{\rho(\vec{r}_O)}{|\vec{r}' - \vec{r}_O|} \right] dV$$

2 ELECTROSTATICS PART 2

The flux of a field across a surface is related to how much and how the field crosses the surface.

$$\Phi_{\vec{B}} = \int_S \vec{B} \cdot \vec{u}_N \, dS$$

Where $\vec{B}$ is the field, $\vec{u}_N$ is the normalized normal vector of the surface, and S is the surface area.

When the surface is flat and the field is uniform:

$$\Phi_{\vec{B}} = \vec{A} \cdot \vec{B}$$

Where $\vec{A}$ is the normalized surface normal multiplied by the surface area, and $\vec{B}$ is the field.

2.1 GAUSS LAW

For a charge INSIDE a CLOSED surface generating an electrostatic field,

$$\Phi(\vec{E}) = \oint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$

However, if the charge is outside,

$$\Phi(\vec{E}) = 0$$

And if there are multiple charges INSIDE the closed surface,

$$\Phi(\vec{E}_{TOT}) = \frac{Q_{TOT}}{\varepsilon_0}$$

2.2 HOLLOW SPHERE

For a hollow sphere of radius R, with a uniform charge on the surface, at a distance r from the centre of the sphere:

When $r \leq R$:

$$Q_{Inner} = 0$$
$$E = 0$$
$$V = k\frac{Q}{R}$$

When $r > R$:

The sphere acts like a point charge at its centre.

$$E = k\frac{Q}{r^2}$$
$$V = k\frac{Q}{r}$$

2.3 INFINITE WIRE AND A CYLINDER

Assume an infinite wire (cylinder) with charge density $\lambda = \frac{dq}{dL}$. Then we have a cylinder of radius r and length h with central axis aligned with the wire. $\therefore Q = \lambda h$

On the top and bottom lids, flux is 0.

On the side walls:

$$\Phi(\vec{E}) = E(2\pi rh)$$
$$\vec{E} = 2k\frac{\lambda}{r}\vec{u}_r$$
$$V_{es} = -2k\lambda \ln r + \text{const}$$

2.4 INFINITE PLANE

Assume an infinite plane with charge density σ (measured in Cm^{-1}), with a gaussian surface being a cylinder cutting perpendicularly through the plane.

On the side walls, flux is zero.

Flux on the taps is:

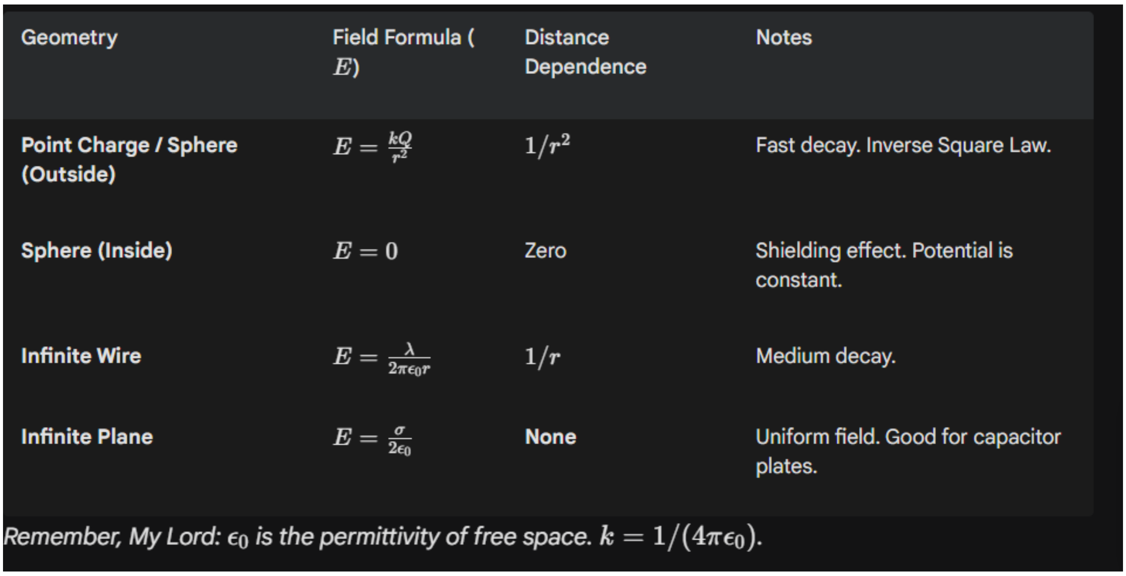
$$\Phi(\vec{E}) = 2ES = \frac{\sigma S}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

And the charge inside the cylinder is σS

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\vec{u}_\perp$$
$$\therefore V_{es} = 0$$

This means that the field does not change with distance.

2.5 SUMMARY



Geometry	Field Formula (E)	Distance Dependence	Notes
Point Charge / Sphere (Outside)	$E = \frac{kQ}{r^2}$	$1/r^2$	Fast decay. Inverse Square Law.
Sphere (Inside)	$E = 0$	Zero	Shielding effect. Potential is constant.
Infinite Wire	$E = \frac{\lambda}{2\pi\epsilon_0 r}$	$1/r$	Medium decay.
Infinite Plane	$E = \frac{\sigma}{2\epsilon_0}$	**None**	Uniform field. Good for capacitor plates.

Remember, My Lord: ϵ_0 is the permittivity of free space. $k = 1/(4\pi\epsilon_0)$.

3 NABLA OPERATOR AND GAUSS LAWS

3.1 THE SOLID ANGLE

The solid angle Ω is the 3D "opening" of a cone with its vertex at a point O and its base defined by a surface A. It is measured in steradians (sr)

$$\Omega = \frac{A}{r^2}$$

For a WHOLE sphere with O being at its centre,

$$\Omega = \frac{4\pi r^2}{r^2} = 4\pi$$

The formula

$$\Omega = 4\pi$$

Can be used for any closed surface where O is inside it.

If the point O is outside the closed surface, then

$$\Omega = 0$$

In general, we also consider the alignment between the surface normal and the offset of the surface point from O, therefore leading to the generalization:

$$\Omega = \int_S \frac{u_n \cdot u_r}{r^2}\, dS$$

Where:

u_n is the normal unit vector of the surface

u_r is the normalized unit vector of the offset from O to dS

3.2 NABLA OPERATOR (∇) AND GRADIENT

For a function

$$f(x_1, x_2 \dots x_n)$$

The Nabla operator defines the vector of most change for f:

$$\nabla f = \begin{pmatrix} \frac{\delta f}{\delta x_1} \\ \frac{\delta f}{\delta x_2} \\ \vdots \\ \frac{\delta f}{\delta x_n} \end{pmatrix}$$

Iff $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is in cartesian coordinates. If it uses other coordinates, such as polar, spherical, cylindrical, etc, then it must be adjusted such that δx_i is scaled correctly.

An example of when ∇ is used is:

$$\vec{F} = -\nabla E_{pot}$$

Cartesian $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$:

$$\nabla f = \begin{pmatrix} \frac{\delta f}{\delta x} \\ \frac{\delta f}{\delta y} \\ \frac{\delta f}{\delta z} \end{pmatrix}$$

Cylindrical $(\boldsymbol{r}, \boldsymbol{\theta}, \mathbf{z})$:

$$\nabla f = \frac{\delta f}{\delta r}\vec{u}_r + \frac{1}{r}\frac{\delta f}{\delta \theta}\vec{u}_\theta + \frac{\delta f}{\delta z}\vec{k}$$

Spherical $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\phi})$:

$$\nabla f = \frac{\delta f}{\delta r}\vec{u}_r + \frac{1}{r}\frac{\delta f}{\delta \theta}\vec{u}_\theta + \frac{1}{r\ \sin\theta}\frac{\delta f}{\delta \phi}\vec{u}_\phi$$

Note: θ is the angle from the positive z-axis (elevation), while ϕ is the angle around the z-axis from the x-axis (azimuth).

For a three-dimensional field $\vec{A}$,

$$\nabla \times A$$

Gives the curl. If curl is zero, then the field is conservative.

The above is just notation to make it look cleaner. The actual meaning is:

$$\nabla \times A = \begin{Bmatrix} \frac{\delta}{\delta x} \\ \frac{\delta}{\delta y} \\ \frac{\delta}{\delta z} \end{Bmatrix} \times \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} = \begin{Bmatrix} \frac{\delta A_z}{\delta y} - \frac{\delta A_y}{\delta z} \\ \frac{\delta A_x}{\delta z} - \frac{\delta A_z}{\delta x} \\ \frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y} \end{Bmatrix}$$

Which is notation magic that uses cross product between an operator and a vector to apply the operator in a particular manner. The dot product variation (shown below) is a little easier.

3.3 DIVERGENCE

$$div(\vec{A}) = \nabla \cdot \vec{A} = \frac{\delta A_{x_1}}{\delta x_1} + \frac{\delta A_{x_2}}{\delta x_2} + \cdots + \frac{\delta A_{x_n}}{\delta x_n}$$

$\nabla \cdot \vec{A}$ represents the “flow rate” of a vector field $\vec{A}$. The output is a scalar.

Cartesian $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$:

$$\nabla \cdot \vec{A} = \frac{\delta A_x}{\delta x} + \frac{\delta A_y}{\delta y} + \frac{\delta A_z}{\delta z}$$

Cylindrical $(\boldsymbol{r}, \boldsymbol{\theta}, \mathbf{z})$:

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\delta(rA_r)}{\delta r} + \frac{1}{r}\frac{\delta A_\theta}{\delta \theta} + \frac{\delta A_z}{\delta z}$$

Spherical $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\phi})$:

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\delta(r^2 A_r)}{\delta r} + \frac{1}{r\sin\theta}\frac{\delta(A_\theta \sin\theta)}{\delta\theta} + \frac{1}{r\sin\theta}\frac{\delta A_\phi}{\delta\phi}$$

Note: θ is the angle from the positive z-axis (elevation), while ϕ is the angle around the z-axis from the x-axis (azimuth).

3.4 DIVERGENCE THEOREM (GAUSS' THEOREM)

$$\oint_S \vec{A} \cdot \vec{u}_n \, dS = \int_V \left(\nabla \cdot \vec{A}\right) dV$$

3.5 GAUSS LAW (DIFFERENTIAL FORM)

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

Where ρ is the charge density measured in Cm^{-3}

The meaning of this:

$\nabla \cdot \vec{E} > 0$: The point is a source (+ve charge). Flux goes out.

$\nabla \cdot \vec{E} < 0$: The point is a sink (-ve charge). Flux goes in.

$\nabla \cdot \vec{E} = 0$: No charge at the point. Solenoidal field.

4 ELECTRIC DIPOLES AND DIELECTRICS

4.1 ELECTRIC DIPOLE MODEL

When two charges $+q$ and $-q$ are separated by a small distance d, the dipole moment is:

$$\vec{p} = q\vec{d}$$

Where $\vec{d}$ is the displacement from the negative charge to the positive. The unit of $\vec{p}$ is Cm

Unlike with point-charges, when far away, potential does not follow $\frac{1}{r}$ proportionality. Instead, it follows a $\frac{1}{r^2}$ proportionality.

$$V \approx k\frac{(\vec{p} \cdot \vec{u}_r)}{r^2}$$

Where $\vec{u}_r$ is the unit vector pointing from the centre of the dipole to the point of observation.

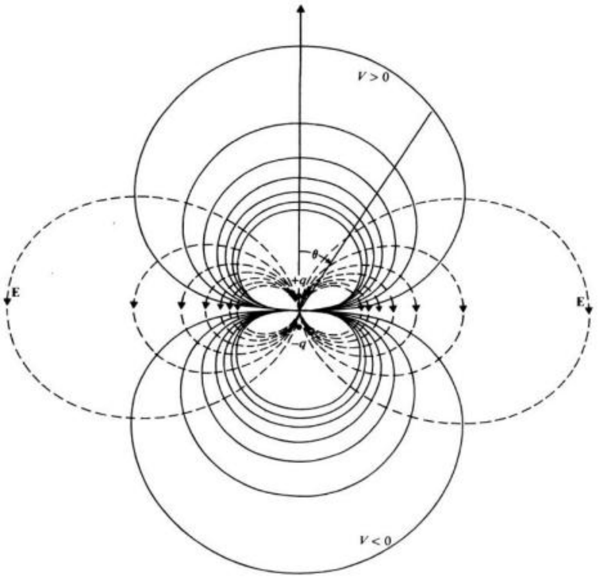
Consequently, the field follows a $\frac{1}{r^3}$ proportionality.

$$\vec{E}_P = k\frac{3(\vec{p} \cdot \vec{u}_r) - \vec{p}}{r^3}$$

And if point P $\vec{r}$ is perpendicular to $\vec{d}$, the equation simplifies to:

$$\vec{E}_{P'} = -k\frac{\vec{p}'}{r^3} = -k\frac{q\vec{d}}{r^3}$$

Note that the field does not point towards or away from the dipole. Instead, in the second example it points downwards.



4.2 DIPOLE IN AN ELECTRIC FIELD

If a dipole is placed in a uniform electric field $\vec{E}_{ext}$, forces are applied.

$$\vec{F}_{net} = 0$$

However, a torque is applied:

$$\vec{\tau} = \vec{p} \times \vec{E}_{ext}$$

The torque aligns the dipole to the field. The dipole is at equilibrium when $\theta = 0$ (stable equilibrium – aligned with field) or when $\theta = 180°$ (unstable equilibrium – anti-aligned with field).

4.3 DIELECTRICS AND POLARIZATION

When a neutral atom is in an electric field, the nucleus (positive) will experience a force opposite to the electrons (negative), therefore creating a dipole moment. The magnitude of the dipole moment depends on the strength of the field and the properties of the atom (orbitals, number of protons/electrons, other).

We call this interaction “Polarization”.

For small intensities of the field (we consider small intensities since we can assume that the effect of the field on the magnitude of the dipole moment can be approximated as linear), we can approximate the dipole moment using Maclaurin expansion:

$$\vec{p} = \varepsilon_0 \alpha \vec{E} + o(\vec{E})$$
$$[\alpha] = C^2 m N^{-1}$$

When we have n identical atoms per unit volume, we can describe the phenomenon with the total dipole moment (aka “Polarization”) $\mathcal{P}$:

$$\mathcal{P} = n\vec{p} = \varepsilon_0 n \alpha \vec{E} = \epsilon_0 \chi \vec{\mathrm{E}}$$

Where $\chi = n\alpha$ is the electrical susceptibility of the material and is determined by the properties of the atoms and the density.

Dielectric strength is the maximum field the material can withstand before breaking down. After such limit, the material will become conductive (gases will become plasma).

4.4 POLARIZATION CONTRIBUTION TO A FIELD

When a material is polarized, it will contribute to the total electric field.

Assume a slab of neutrally charged material is placed in a uniform electric field $\vec{E}_0$ (two of the faces each with surface area S are aligned with the field, and the thickness of the slab is d). It will therefore have a polarization of $\mathcal{P} = \epsilon_0 \chi \vec{E}_0$. While the overall solid will be neutral, but the surfaces pointing towards and against the field will be charged.

The net field will be:

$$\vec{E}_{NET} = \vec{E}_0 + \vec{E}_P$$

Or since the field caused by polarization is opposite to E_0:

$$E_{NET} = E_0 - E_P$$

$$E_P = \frac{\sigma_P}{\varepsilon_0} = \frac{Q_P}{S\varepsilon_0}$$

From the dipole moment formula, we can achieve a formula for total dipole moment in a material:

$$P = Q_P d = \mathcal{P}V$$

Or

$$P = \sigma_P SD = \mathcal{P}Sd \Rightarrow \mathcal{P} = \sigma_P$$

$$\therefore E_P = E_0 - \frac{\mathcal{P}}{\varepsilon_0}$$

And for small fields

$$E = \frac{E_0}{1+\chi}, \qquad E_0 = E(1+\chi)$$

$\epsilon_r = \kappa = 1 + \chi$ is called the relative dielectric constant of the material, or relative permittivity.

Dielectric constant of the material:

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

4.5 ELECTRICAL DISPLACEMENT VECTOR

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{\mathcal{P}}$$
$$[D] = Cm^{-2}$$

Or for weak fields:

$$\vec{D} \approx \varepsilon \vec{E}$$

This value remains the same for all fields when going through one median to another, assuming no free charge on any surface.

This allows for Gauss law to be rewritten as:

$$\oint_S \vec{D} \cdot \vec{u}_n \, dS = Q_{FREE}$$

The concept of the electrical displacement vector helps define what happens when a field goes from median 1 to median 2. We separate the field into two components: $E_{\parallel}$ being the component parallel (tangent) to the surface, and $E_{\perp}$ being the component perpendicular (normal) to the surface.

$$E_{1,\parallel} = E_{2,\parallel}$$

$$D_{1,\perp} = E_{2,\perp} \Rightarrow \varepsilon_1 E_{1,\perp} = \varepsilon_2 E_{2,\perp}$$

This causes a refraction of the field across medians since one component remains unchanged while the other changes.

5 CONDUCTORS

The model that describes conductors is based on two assumptions:

- The electrons within (and added if any) within the material cannot leave from the surface.
- The electrons within (and added if any) can move freely within the material.

For a material to be neutral, the total negative charge must be EXACTLY the same as the total positive charge.

If the conductor is in static equilibrium, then there must be no net electric field within the material, or the electrons would move, leading to the condition to not being static (out of equilibrium situation).

5.1 CONDUCTOR IN AN ELECTRIC FIELD

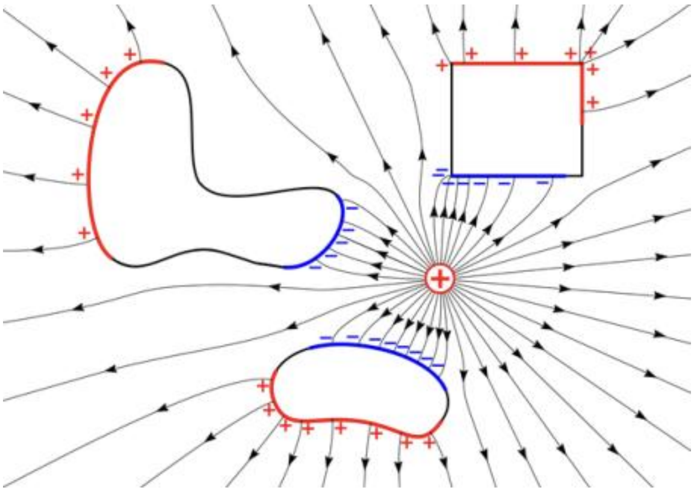
If the material is placed in an electric field, the conductor must contribute itself to the field to neutralize it and reach static equilibrium. The field $\vec{E}_{EXT}$ causes electrons to move in the opposite direction of the field, causing a negative charge on one side and a positive charge on the other, and therefore a field $\vec{E}_{INT}$ is induced, leading to the static equilibrium where in every point inside the conductor:

$$\vec{E}_{Net} = \vec{E}_{EXT} + \vec{E}_{INT} = 0$$

Or

$$\vec{E}_{EXT} = -\vec{E}_{INT}$$

The charge which is forced to the surface is called induced charge.



5.2 CHARGED CONDUCTORS

If a charge is added to a conductor (via adding a huge number of electrons), the electrons will internally repel, and since they cannot leave the material, they move to the surface. They will spread out to form zero field everywhere inside the material (minimum potential is reached).

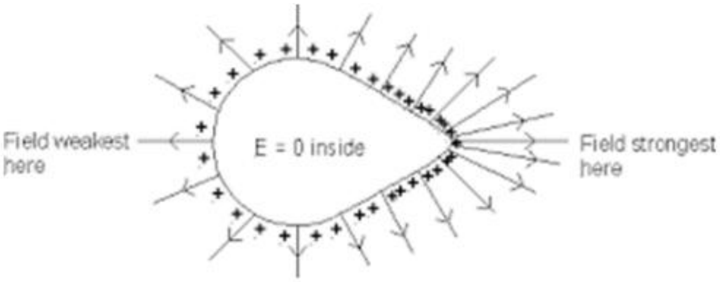
Charges tend to move towards tips of the conductive object if one or more are present.

Close to a charged conductor, the electrostatic field is perpendicular to the surface. If this was false, charges on the surface would move (no static equilibrium).

Columb's Law states that for a charged conductor with surface charge density of σ, the electric field close to the surface (outside the material) is

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\vec{u}_n$$

Since charges tend to accumulate on tips, more curvature means a higher surface density and therefore a stronger field. (electric field is stronger at the tips)



5.3 HOLLOW CONDUCTORS AND SHIELDING

Electric field inside the cavity of a hollow conductor must also be $\vec{0}$ in a stable equilibrium, just like for solid ones. (for a hollow conductor in a field with no charges in the cavity)

If a charge $Q > 0$ is placed inside the hollow conductor, then electrons will move to the inner surface, and the total charge of the inner surface will be equal to $-Q$. Therefore, the charge of the outer surface will be equal to Q. In this case, there will be a charge in the hollow section, going from Q to the surface, and the density depends on the position of Q and on the shape of the cavity. (Electrons will bunch up near the charge if Q is closer to one part of the surface than another.)

The outside of the conductor will have a field perpendicular to the surface.

Faraday rooms (and cages) use this to create a space where no external field can penetrate through. An example is a car.

When charges are poured into a hollow conductor, field inside is zero everywhere, and therefore potential difference between any two points is zero, or all points have equal potential.

$$\Delta V = 0$$

5.4 CAPACITANCE

$$C = \frac{Q}{V}$$

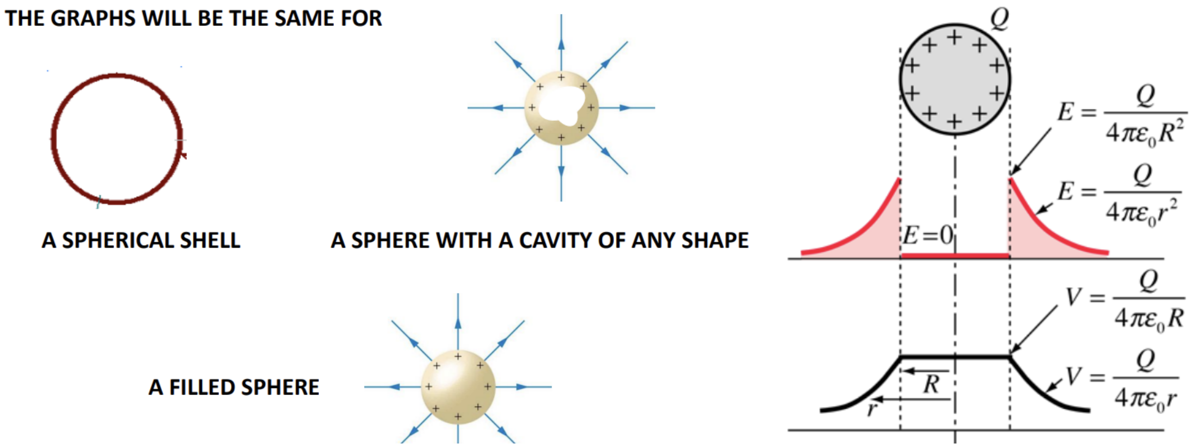
(Unit is F or Farad)

For a solid conductive sphere of radius R and charge Q,

$$V = k\frac{Q}{R} \Rightarrow C = \frac{R}{k}$$

Capacitance gives information about how much potential is generated by adding a certain charge to a conductor.

One farad is huge. For instance, if the Earth was perfectly conductive, it would have a capacitance of about $0.71mF$



When two charged spherical conductors of charges Q_1, Q_2 and radii R_1, R_2 are connected via a wire of negligible diameter, the potential difference will cause charge to flow from one to the other.

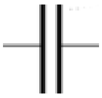
Equilibrium will be achieved when the potentials of the two conductors are the same.

$$V_1 = V_2 \Rightarrow \frac{Q_1}{C_1} = \frac{Q_2}{C_2} \Rightarrow \frac{Q_1}{R_1} = \frac{Q_2}{R_2}$$

Above results were explained with spheres, but the same rules apply to all shapes and sizes:

- $C = \frac{Q}{V}$
- C depends only on shape and size of the conductor
- Charge between two connected conductors splits following their capacitances

6 CAPACITORS



When two electrodes of opposite charge are placed very close, their respective charges tend to clump up closest to the other electrode, forming a capacitor. Total charge for a capacitor is always zero.

$$C = \frac{Q}{\Delta V}$$

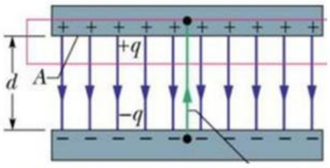
For a capacitor we have:

$$Q = Q_+ = -Q_-$$
$$\Delta V = V_+ - V_-, \qquad V_+ > 0 > V_-$$

C of a capacitor will be determined by the shape and the relative positions of the electrodes.

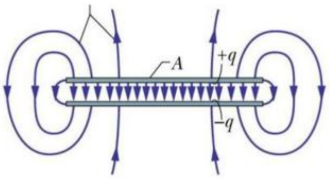
Parallel-plate capacitors are the simplest type of capacitor, formed of two parallel plates placed very close to each other, where the distance between them is negligible with respect to their surface area.

IDEAL



- UNIFORM FIELD BETWEEN THE PLATES
- NO FIELD OUTSIDE

REAL



- NON-UNIFORM FIELD (ESPECIALLY AT EDGES)
- FIELD 'LEAKS' OUTSIDE

Outside the capacitor,

$$E = 0$$

Between the plates of area A,

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$

And the direction is from the positive electrode to the negative, perpendicularly to the surfaces.

$$\Rightarrow \Delta V = \frac{Qd}{\varepsilon_0 A}, \qquad C = \frac{\varepsilon_0 A}{d}, \qquad U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\mathrm{C}\Delta V^2 = \frac{1}{2}Q\Delta V$$

This is all assuming the capacitor is ideal.

$$U = \int_{VOL} \frac{1}{2}\varepsilon_0 E^2 d\mathcal{V}$$

Where VOL is the volume where $E \neq 0$ and $d\mathcal{V}$ is an infinitesimal volume.

In a capacitor, V can be called “voltage drop”.

If we have a dielectric (insulator) between the electrodes, we get:

$$C = \frac{\varepsilon A}{d}$$

In dielectrics, $\varepsilon > \varepsilon_0$, therefore capacitance is increased with a dielectric between electrodes.

To maximize the capacitance, we can therefore:

- Insert a dielectric with very high relative dielectric constant
- Increase electrode area by rolling the plates
- Reduce space between the plates (By having a very thin dielectric)

If we charge a capacitor, and then vary its capacitance, we have two cases –

Constant potential difference:

$$\Delta U = \frac{1}{2}\Delta C \Delta V^2$$

Constant charge:

$$\Delta U = \frac{1}{2}\Delta\left(\frac{1}{C}\right)Q^2$$

Note that if we increase C with constant ΔV, $\Delta U > 0$, while if Q is constant $\Delta U < 0$

If we place several capacitors in parallel, they will be equivalent to a single capacitor with:

$$C_{eq} = \Sigma C_i$$

While in series:

$$\frac{1}{C_{eq}} = \Sigma \frac{1}{C_i}$$

7 ELECTRICAL RESISTANCE

7.1 CURRENT DENSITY

$$\vec{J} = qn\vec{v}$$

Where:

$\vec{J}$ is the current density
q is the charge of one carrier ($1.6 \times 10^{-19}C$ for electrons)
n is the number of carriers per m^3, or charge carrier density
$\vec{v}$ is the electron drift speed.

Current density is a vector that describes current flow per unit area.

We can use this to calculate current:

$$I = \frac{dQ}{dt} = \int_S \vec{J} \cdot \vec{u}_N \, dS$$
$$[I] = A = Cs^{-1}$$

Current is positive if it leads to an increase in current downstream.

7.2 RESISTIVITY AND RESISTANCE

The model for ideal conductors states that electrons are free to move. However, this would lead to an electric field acting on a wire of infinite length to contain electrons that infinitely accelerate, creating increasingly more current over time. We know that this is not the case.

This is because charges lose energy when colliding or interacting with the material.

We assume that a charge will collide every 2τ seconds (τ is called the lifetime). When colliding, all the kinetic energy will be transferred from the electron to the material. Therefore, an electron will collide, accelerate for 2τ seconds, collide and therefore speed is "reset", and repeat.

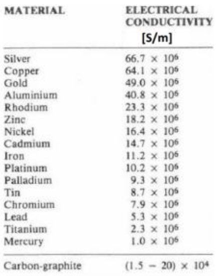
$$v_{MAX} = a(2\tau) = \frac{q}{m}E(2\tau)$$
$$\Rightarrow \vec{v}_{AVG} = \frac{q}{m}\vec{E}\tau$$

$$\vec{J} = qn\vec{v} = qn \cdot \frac{q}{m}\vec{E}\tau = \frac{q^2 n\tau}{m}\vec{E} = \sigma\vec{E}$$

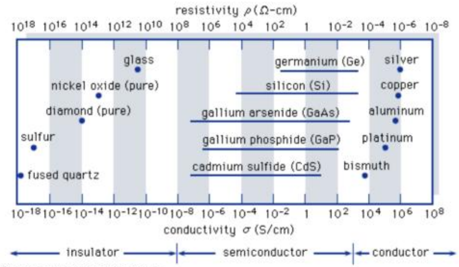
Where σ is the electrical conductivity of the material.

$$\rho = \frac{1}{\sigma}$$

ρ is the electrical resistivity of the material.



MATERIAL	ELECTRICAL CONDUCTIVITY [S/m]
Silver	66.7×10^6
Copper	64.1×10^6
Gold	49.0×10^6
Aluminium	40.8×10^6
Rhodium	23.3×10^6
Zinc	18.2×10^6
Nickel	16.4×10^6
Cadmium	14.7×10^6
Iron	11.2×10^6
Platinum	10.2×10^6
Palladium	9.3×10^6
Tin	8.7×10^6
Chromium	7.9×10^6
Lead	5.3×10^6
Titanium	2.3×10^6
Mercury	1.0×10^6
Carbon-graphite	$(1.5 - 20) \times 10^4$



© 2004 Encyclopædia Britannica, Inc.

For a uniform wire of length d with cross-sectional area S:

$$
\begin{aligned}
I &= \sigma E S \\
\Delta V &= E d \\
\Delta V &= \frac{d}{\sigma S} I = \frac{d\rho}{S} I \\
I &= \frac{\sigma S}{d} \Delta V = \frac{S}{\rho d} \Delta V
\end{aligned}
$$

$$
\begin{aligned}
R &= \frac{d}{\sigma S} = \frac{d}{S}\rho \\
\Delta V &= IR
\end{aligned}
$$

(First Ohm's Law)

$$[R] = \Omega = VA^{-1}, \qquad [\rho] = \Omega m, \qquad [\sigma] = \Omega^{-1} m^{-1}$$

Electrical conductance:

$$
\begin{aligned}
G &= \frac{1}{R} \\
[G] &= V^{-1}A = S
\end{aligned}
$$

The unit of G is called a Siemen.

7.3 HEATING EFFECT OF RESISTORS

Because of the collisions between the material and electrons, thermal energy is generated. When the "thermal steady state" is reached, temperature will not increase further: heat emitted by the resistor is equal to the heat generated by the current.

If we consider a resistor having a potential of V_+ on one side and V_- on the other, we can consider that for every electron, the change in electrostatic potential energy will be:

$$\Delta U = q(V_+ - V_-)$$

Therefore, the power released to the resistor is:

$$P = I\Delta V = RI^2 = \frac{(\Delta V)^2}{R}$$

The time taken to reach the DC situation once the potential is very short (often microseconds or less).

Resistance varies with temperature!

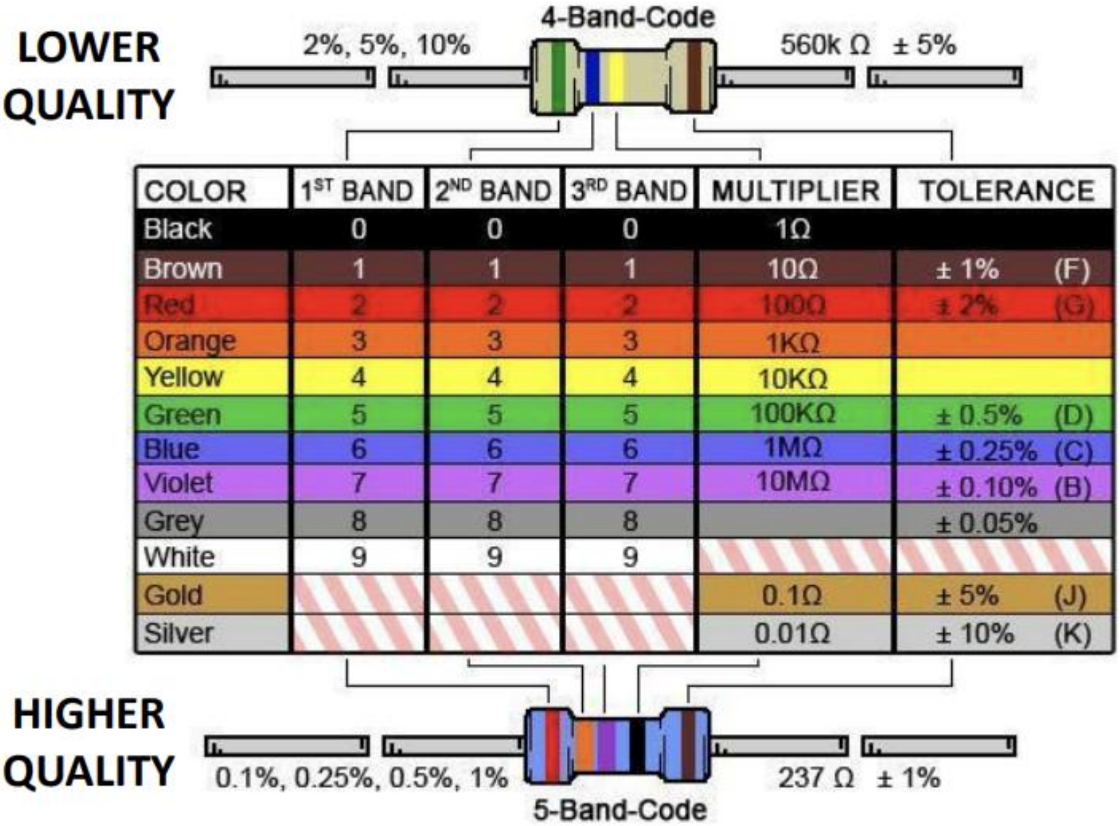
7.4 FIRST KIRCHHOFF'S LAW

At a node (a point where 3 or more wires intersect):

$$\Sigma I = 0$$

Current entering the node is positive, current leaving is negative (convention). This law is a consequence of charge conservation.

7.5 RESISTORS



COLOR	1^{ST} BAND	2^{ND} BAND	3^{RD} BAND	MULTIPLIER	TOLERANCE
Black	0	0	0	1Ω	
Brown	1	1	1	10Ω	± 1% (F)
Red	2	2	2	100Ω	± 2% (G)
Orange	3	3	3	1KΩ	
Yellow	4	4	4	10KΩ	
Green	5	5	5	100KΩ	± 0.5% (D)
Blue	6	6	6	1MΩ	± 0.25% (C)
Violet	7	7	7	10MΩ	± 0.10% (B)
Grey	8	8	8		± 0.05%
White	9	9	9		
Gold				0.1Ω	± 5% (J)
Silver				0.01Ω	± 10% (K)

For resistors in series:

$$R_{TOT} = \Sigma R_i$$

In parallel:

$$\frac{1}{R_{TOT}} = \Sigma \frac{1}{R_i}$$

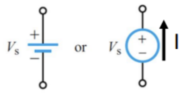
8 DC GENERATORS

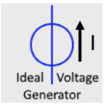
Active elements are electrical components that provide electrical power to the circuit. They are usually either batteries (DC) or generators (AC or DC).

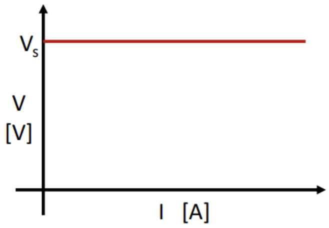
An ideal DC voltage generator keeps the same potential difference between the two terminals constant, irrespective of the components connected and time.

When an electron enters the circuit with potential V_- and exits with potential V_+,

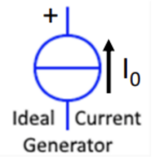
$$\Delta E_q = q(V_+ - V_-) = qV_s$$

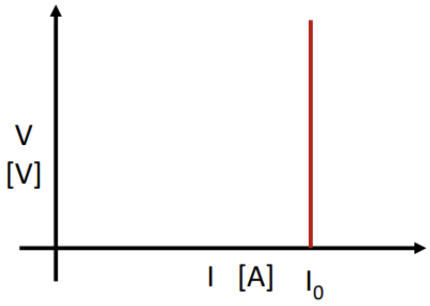






An ideal DC current generator always provides the same current irrespective of components and time.





8.1 SECOND KIRCHHOFF'S LAW

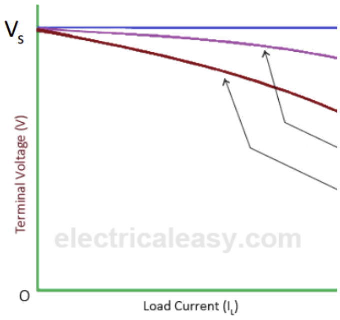
The sum of all potential differences around a closed loop is always zero.

$$\Sigma V = 0$$

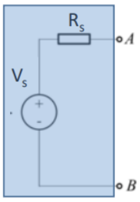
This is because electrostatic force is conservative.

8.2 REAL GENERATORS

An ideal generator assumes 100% efficiency in the energy conversion to electrical and ability the maintain their quantities. Real generators do not follow either: they are not perfectly efficient and as current load increases, voltage tends to decrease.



We can therefore describe a real voltage generator as a series of an ideal generator and a resistor:



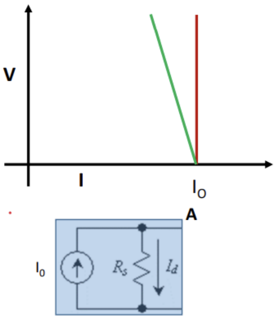
Note: This is not reality, this is just an EQUIVALENT circuit. More advanced equivalent circuits also consider the non-linearity of the IV relation.

$$V = V_S - aI$$
$$[a] = \Omega$$

For real current generators, we label I_0 the current provided when the terminals are short-circuited. When components are connected, the voltage across the terminals decreases as load increases

$$I = I_0 - bV$$
$$[b] = \Omega^{-1}$$

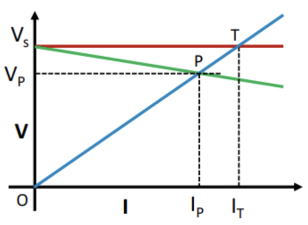
We can describe a real current generator as an ideal generator and a resistor in parallel.



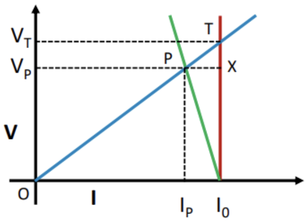
REAL GENERATORS GRAPHS

IN AN I-V GRAPH **THE AREAS REPRESENT A POWER**. IN LIGHT OF THIS, WE WILL DISCUSS THE POWER ISSUED IN A CIRCUIT WITH REAL GENERATORS

VOLTAGE GENERATOR



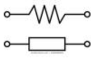
CURRENT GENERATOR



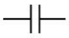
RECTANGLE V_S-T-I_T-O POWER DELIVERED BY AN **IDEAL GENERATOR** TO AN EXTERNAL RESISTOR (**WORKING POINT T**) **RECTANGLE V_T-T-I_O-O**

RECTANGLE V_P-P-I_P-O POWER DELIVERED BY A **REAL GENERATOR** TO AN EXTERNAL RESISTOR (**WORKING POINT P**) (I_P, V_P) **RECTANGLE V_P-P-I_P-O**

RESISTOR



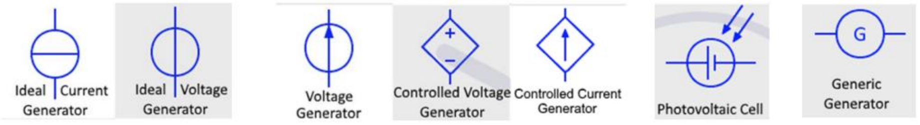
$\Delta V = R\,I$



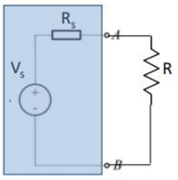
CAPACITOR

$Q = C\,\Delta V$

GENERATORS



REAL DC VOLTAGE GENERATOR



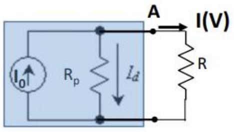
$$\Delta V_{AB} = V_S \frac{R}{(R_S+R)} = V_S \frac{1}{\left[1+\left(\frac{R_S}{R}\right)\right]}$$

R_S: THE SMALLER THE BETTER

REAL DC CURRENT GENERATOR

$$I(V) = I_0 \frac{R_p}{R+R_p} = I_0 \frac{1}{\left[1+\left(\frac{R}{R_p}\right)\right]}$$

R_P: THE LARGER THE BETTER

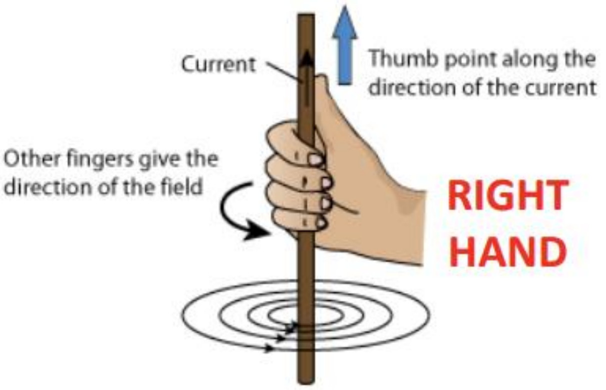


9 MAGNETOSTATICS

Lorentz Force:

$$\vec{F} = q\vec{v} \times \vec{B}$$
$$\Rightarrow \vec{a} = \frac{q}{m}(\vec{v} \times \vec{B})$$

Where $\vec{B}$ is called the magnetostatic field. (Note: × implies cross product)



Notation for the field:

⊙

$\vec{B}$ EXITS THE SCREEN

⊗

$\vec{B}$ ENTERS THE SCREEN

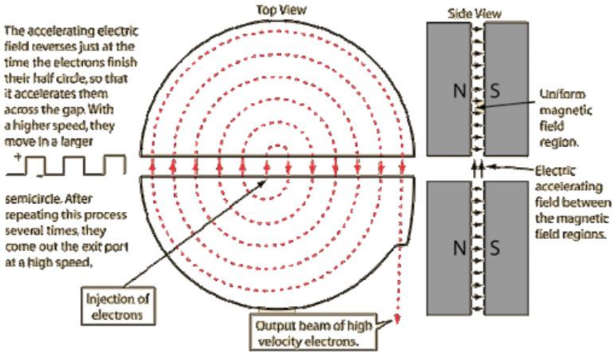
Because the acceleration is always perpendicular to velocity, in an isolated environment with a charge and a uniform magnetic field, if the charge is moving, it will move in circular motion, with radius, period and angular speed:

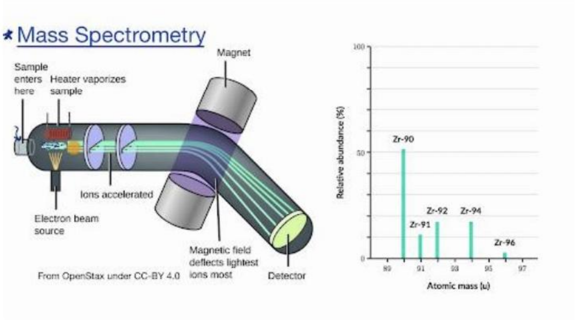
$$R = \frac{mv}{|q|B}$$
$$T = \frac{2\pi m}{|q|B}$$
$$\omega = \frac{|q|B}{m}$$

The Lorentz force performs NO WORK. Therefore:

$$P = \vec{F} \cdot \vec{v} = q(\vec{v} \times \vec{B}) \cdot \vec{v} = 0$$

Since $\vec{v} \times \vec{B} \perp \vec{v} \Rightarrow (\vec{v} \times \vec{B}) \cdot \vec{v} = 0$





From OpenStax under CC-BY 4.0

(Left: Cyclotron – device used to accelerate charged particles)

If a wire with DC current is in a magnetic field, a force is applied to the wire:

$$\overrightarrow{dF} = I\left(\overrightarrow{dL} \times \vec{B}\right)$$

Where $\overrightarrow{dF}$ is the force applied to an infinitesimal piece of wire of length (and direction) $\overrightarrow{dL}$. Therefore, the total force is:

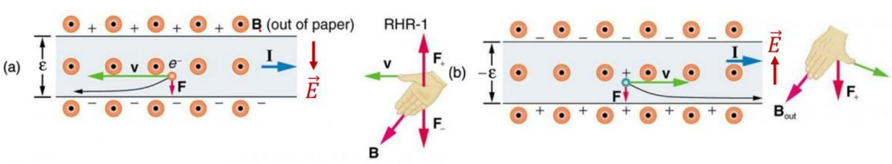
$$\vec{F} = \int_L I\left(\overrightarrow{dL} \times \vec{B}\right)$$

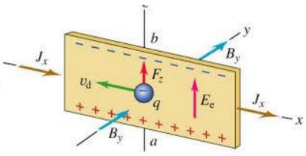
Which for a straight wire of length L,

$$\vec{F} = I\left(\vec{L} \times \vec{B}\right)$$

9.1 HALL EFFECT

For a piece of wire with DC current going through it (in this case assume it is to the right), if it is in a uniform magnetostatic field perpendicular to the wire (in this case assume going out of the page), then the charged particles carrying the current (usually electrons) will be pushed downwards. Since they cannot escape the material, they bunch up at the bottom of the wire, creating a potential difference ϵ between the top and bottom of the material. However, the sign of this will depend on the sign of the charge of the carrier. For electrons, the electric field will point downwards, while if the carrier was positive, it would point upwards.





9.2 MAGNETIC DIPOLES

Magnetic needles (often found in compasses) are ferromagnetic pins that try to align (parallel to) with a magnetic field if it is in one. They do not undergo translation, just rotation.

These needles can be called magnetic dipoles: two rigidly connected charges of equal magnitude and opposite sign. Each side of the pin is a pole. It has a property called magnetic dipole moment, with symbol $\vec{\mu}$ or sometimes $\vec{m}$, which depends on the material of the dipole and on the mass.

Magnetic dipoles behave similarly to electric dipoles in electric fields: they try to align with the magnetic field.

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$
$$\vec{U} = -\vec{\mu} \cdot \vec{B} = -\mu B \cos \theta$$

9.3 GENERATING A MAGNETIC FIELD

Magnetic fields (just like electric fields) are generated by charges.

If we consider that:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \vec{u}_r$$
$$\vec{B} = \frac{\mu_0}{4\pi} \frac{Q(\vec{v} \times \vec{u}_r)}{r^2} = \varepsilon_0 \mu_0 \left(\vec{v} \times \vec{E}\right) = \frac{\vec{v} \times \vec{E}}{c^2}$$
$$\mu_0 = 4\pi \times 10^{-7} TmA^{-1}$$
$$[B] = T$$

Where T is the unit Tesla and c is the speed of light in a vacuum.

It is therefore proportional to (in magnitude) both electric field and velocity of the charges that generate it, and perpendicular to both.

If we have two charges (q and Q) moving with equal velocity $\vec{v}$ (assume right. If we analyse the forces on q due to Q, we can observe that the $\vec{F}_E$ points downwards and $\vec{F}_B$ points upwards, and the ratio of their intensities is:

$$\frac{F_B}{F_E} = \frac{v^2}{c^2}$$

Since $v < c$, The ratio will always be smaller than 1, and unless the speed of the charges approaches the speed of light, the force due to the magnetic field will be negligible with respect to the force due to the electric field.

In a wire with DC current flowing, the wire is neutral and charges are moving, therefore only a magnetic field is present (No electric field).

Laplace's Law/Biot-Savart 1st Law:

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\overrightarrow{dL} \times \vec{u}_r}{r^2}$$

Where:
$d\vec{B}$ is the magnetic field due to by an infinitesimal piece of wire with length (and direction) $\overrightarrow{dL}$ and current I at a displacement of $\vec{r}$ from the piece.

Since we cannot have a current flowing across ONLY an infinitesimal wire, we need to consider the whole circuit (with path Γ), which is given by the **Biot-Savart 2nd Law:**

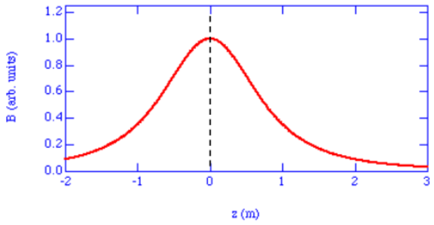
$$\vec{B}(P) = \oint_\Gamma \frac{\mu_0}{4\pi} \frac{I\overrightarrow{dL} \times \vec{u}_r}{r^2} = \frac{\mu_0}{4\pi} I \oint_\Gamma \frac{\overrightarrow{dL} \times \vec{u}_r}{r^2} = \mathrm{I}\xi(\mathrm{P})$$

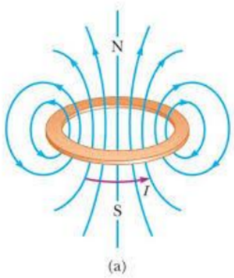
If the circuit is a closed loop (circle), we can evaluate the magnetic field along the axis of the circle:

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{IS}{(a^2 - x^2)^{\frac{3}{2}}} \vec{u}_x$$

Where:

- I is the current going through the loop
- S is the area of the loop $S = \pi a^2$
- a is the radius of the loop
- $\vec{x}$ is the position vector of the point we measure at (along the axis of the loop)





The loop behaves similarly to a magnetic pin (or bar magnet), so we can give it a magnetic dipole moment:

$$\vec{\mu} = IS\vec{u}_\mu$$

Where $\vec{u}_\mu$ is the direction, which is perpendicular to the surface of the loop, and the verse is given by the current direction. (Right hand rule)

9.4 AMPERE LAW

The line integral of a magnetostatic field along any closed line (Amperian loop Γ_A) is proportional to the intensity of the current flowing through the surface delimited by the line.

$$\oint_{\Gamma_A} \vec{B}(\vec{r}) \cdot d\vec{s} = \mu_0 I$$

And if multiple circuits are present, we can use:

$$\oint_{\Gamma_A} \vec{B}_{TOT}(\vec{r}) \cdot d\vec{s} = \mu_0 \Sigma I_i$$

While only considering wires that intersect the surface within the path. I_i is also signed, depending on its direction relative to the surface normal.

If we apply the Ampere Law to a circular path of radius r centred around an infinitely long wire with current I, we get:

$$B(r) = \frac{\mu_0}{2\pi} \frac{I}{r}$$

Where the direction of the field follows the right-hand rule.

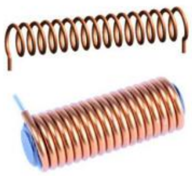
If we have two identical, parallel wires at a distance r with radius negligible with respect to their length, and currents I_1, I_2 flowing through them, then the force applied to a segment of the wire of length L due to the magnetic field of the other is:

$$F = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{r}L$$

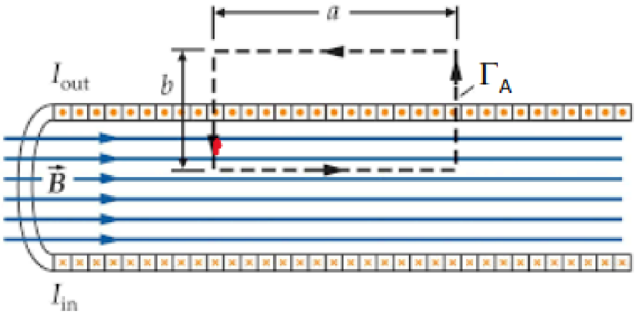
The force is attractive if the currents are in the same direction, and repulsive if currents are in opposite directions.

9.5 SOLENOIDS

A solenoid is simply a sequence of loops, set up like a spring, where current flows through these loops.



We can apply the Ampere Law to a solenoid via the Amperian loop described by the rectangle shown below:



We then get:

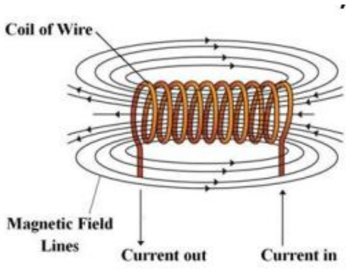
$$\oint_{\Gamma_A} \vec{B}(\vec{r}) \cdot d\vec{s} = \mu_0 n a I$$

Where n is the number of loops.

We can therefore measure the field generated by the solenoid:

$$B_{OUTSIDE} \approx 0$$
$$B_{INSIDE} = \mu_0 n I$$

We consider the field outside to be approximately 0 because it is negligible with respect to the field inside. It is zero for a solenoid of infinite length, but since we can't have that in the real world, we can approximate it to zero, mostly as the length of it increases.



Gauss' Law for magnetic fields states that: the flux of the magnetostatic field through any closed surface is always zero.

$$\oint_S \vec{B}(\vec{r}) \cdot \vec{u}_r dS = 0$$

Using the right-hand rule, the north pole of a magnet (or electromagnet) is indicated by the thumb. A single DC current produces both north and south poles, never just one.

10 MATERIALS IN MAGNETIC FIELDS

There are three types of materials regarding magnetism:

- Diamagnetic materials
  - The material has no intrinsic magnetic dipoles. When placed in a magnetic field, magnetic dipoles are induced and they generate a field opposing the external one.
- Paramagnetic materials
  - The material has randomly oriented magnetic dipoles that do not interact. The presence of an external field leads to dipoles orientations that contribute an additional field with the same orientation of the external field.
- Ferromagnetic materials (and related categories)
  - The material has magnetic dipoles that strongly interact among them, leading to a collective behaviour influenced by the external field.

$$\vec{H} = \frac{\vec{B}}{\mu_0}$$

$\vec{H}$ is called the magnetizing field. $[H] = Am^{-1}$

The magnetization vector $\vec{\mathcal{M}}$ is defined as the magnetic dipole moment per unit volume. For small fields:

$$\vec{\mathcal{M}} \approx n\beta\vec{H} = \chi_m\vec{H}$$

Where:

- β is a value depending on characteristics of dipoles
- n is the number of dipoles per unit volume
- χ_m is the magnetic susceptibility of the material

The total magnetic field inside the material will be:

$$\vec{B} = \mu_0\vec{\mathcal{M}} + \mu_0\vec{H}$$

For small fields:

$$\vec{B} \approx \mu_0(1 + \chi_m)\vec{H}$$

And magnetic permeability of the material:

$$\mu = \mu_0\mu_r = \mu_0(1 + \chi_m)$$
$$[\mu] = [\mu_0] = TmA^{-1} = kgmC^{-2}$$

10.1 DIAMAGNETISM

For diamagnetic substances, $\chi_m \approx -10^{-5}$, meaning that their magnetic permeability is close, but slightly lower, than in a vacuum. In most situation, their diamagnetic property can be ignored (use vacuum instead).

10.2 PARAMAGNETISM

In paramagnetic substances, magnetic dipoles are present even when no external magnetic field is applied.

The dipoles do not interact.

Thermal agitation causes random orientation of dipoles.

Total magnetic field is zero

When an external magnetic field is applied, the dipoles undergo a torque that align them in the direction of the field. This alignment is counteracted by the thermal agitation. When the field is high enough, the thermal agitation is overwhelmed, and the dipoles completely align, so that magnetization saturates.

Curie Law:

$$\chi_m T = \text{const}$$

For paramagnetic materials, $\chi_m \approx 10^{-4}$, meaning that relative magnetic susceptibility is slightly greater than 1. Therefore, the magnetic field inside the material is slightly higher than in a vacuum. We can generally neglect the paramagnetic effect.

When the magnetic field is large, we can no longer use the approximation for the field inside, and we therefore need to use the exact form

$$\vec{B} = \mu_0 \vec{\mathcal{M}} + \mu_0 \vec{H}$$

10.3 FERROMAGNETISM

Is based on a quantum phenomenon called "exchange interaction", where all atoms have their magnetic dipoles, which all interact, leading to a collective behaviour.

The behaviour of a ferromagnetic material is determined by:

- Exchange interaction between neighbouring atoms (keeping dipoles parallel. Energy associated with this interaction is negative and the interaction is short range – in the range of a few nm)
- Interaction with the external field (aligning dipoles parallel to the field and decreasing energy)
- Interaction of every dipole with the magnetic fields of all other dipoles (pushing dipoles to be antiparallel. Much weaker than exchange interaction and has negative energy when dipoles are antiparallel. This is longer range – in the range of the μm)

A magnetic domain is a part of a ferromagnetic material where all dipoles have the same alignment. The domain is usually quite small (0.1-1mm in diameter). Large magnetic domains can create very strong magnetic fields.

Within a domain, exchange interactions prevail, since they are strong and short-range. However, between domains, dipole-dipole interactions prevail since they are long range.

Increasing the size of the domain will result in the positive energy (from dipole-dipole interactions) to increase, while negative energy (from exchange interactions) remains almost the same. This limits the size of domains when there is no external magnetic field, since nature prefers lower energy states.

The transition between domains is through domain walls, which are parts of the material where orientation of dipoles **gradually** changes from one domain to another.

Overall, a large block of ferromagnetic material will have no net magnetic field on its outside, since ones from individual domains cancel out.

If a domain gets too large it will split into two domains with opposing directions. The maximum domain size is given by.

$$L_{dom}^3 = \frac{|e_{exc}|}{|e_{dom}|} L_{exc}^3$$

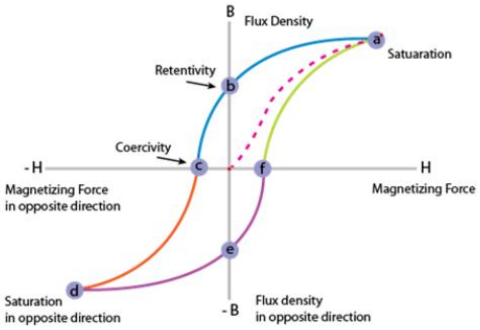
Where:

- L_{dom}^3 is the maximum volume of a domain
- e_{exc} and e_{dom} are the energies per dipole for exchange interaction and dipole-dipole interactions respectively.
- L_{exc} is a property of the material related to domain walls.
- If a magnetic field is applied to the material, dipoles try to align with the field, therefore increasing domain size and increasing alignment of dipoles to the field. With a large enough field, the material becomes one single domain.

For ferromagnetic materials, we cannot use the linear approximation for $\vec{\mathcal{M}}$, therefore we need to use the precise form for B:

$$\vec{B} = \mu_0(\vec{H} + \vec{\mathcal{M}})$$

10.3.1 Hysteresis Loop



The hysteresis loop diagram describes magnetic properties of materials. When a raw ferromagnetic material starts it will be at the origin: no generated magnetic field and no magnetic field applied to it.

a) If you apply a magnetizing field, it will get to point a, with a limit of how magnetic the material can get. At this point, there is one single domain with dipoles aligned with the external field.
b) If the magnetizing field is removed, the material will not go back to point O. It will instead go to point b. This point is quite high in the B axis for materials that can become permanent magnets (they produce a significant magnetic field even with no external field), while it is low for electromagnets.

c) To then fully remove the magnetic field generated by the material, a negative magnetizing field needs to be applied, bringing it to point c.
    a. Low coercivity means that it is easy to remove the magnetic property of a material (eg. Transformer core)
    b. High coercivity means that it is hard to remove the magnetic property of a material (eg. Neodymium magnet)

d) The loop then continues in the opposite direction. The material will not go back to O unless strongly heated.

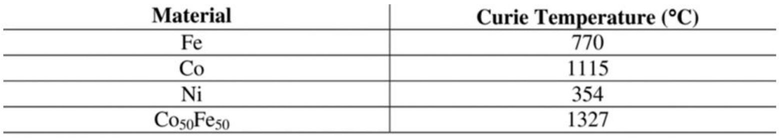
The area within the graph is the energy required for each loop which is dispersed as heat. The wider the graph, the greater the energy.

- High energy is useful for objects that should maintain their magnetic state (eg. Hard drives, bar magnets)
- Low energy is useful for objects that should change magnetic state often (eg. Motors, transformers)

A magnet is a ferromagnetic material where magnetization is still present (with no external field).

To produce a magnet:

1. The material is increased above its Curie temperature
2. It is cooled in absence of a magnetic field
3. External magnetic field is applied up to the point where magnetization is maximized (saturated)
4. Remove the external field.



Material	Curie Temperature (°C)
Fe	770
Co	1115
Ni	354
$Co_{50}Fe_{50}$	1327

Gauss' Law tells us that the magnetic field which is found inside the magnet must also be found outside of it ($B_{IN} = B_{OUT}$)

11 TIME-DEPENDANT FIELDS

11.1 DEPENDANT MAGNETIC FIELD

When the magnetic field is not constant, some rules remain unchanged:

- Lorentz Force
- Magnetic field generated by a moving charge
- Behaviour of dipoles in magnetic fields
- Gauss' Law

However, Ampère's Law does not apply in its Magnetostatic form. We therefore need to correct it for it to apply everywhere (Ampère-Maxwell Law):

$$\oint_{\Gamma_A} \vec{B}(\vec{r}) \cdot d\vec{s} = \mu_0 I + \mu_0 I_d = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi(\vec{E})}{dt}$$

Where I_d is called the "displacement current". The name is misleading since it does not represent any displacement, but a variation of electric flux intersecting the imaginary surface created by the Amperian loop.

The Ampère-Maxwell Law shows that a magnetic field can not only be generated by a moving charge, but also a variation in electric flux over time. (time dependant electric field or observer surface)

The unit for magnetic flux is the Weber:

$$[\Phi(\vec{\mathrm{B}})] = Wb$$

11.2 DEPENDANT ELECTRIC FIELD

If an electric field is not uniform, Gauss' law still applies:

$$\oint_{\Gamma} \vec{E}(\vec{r}) \cdot \vec{u}_N \, dS = \frac{Q}{\varepsilon_0}$$

However, we need to modify Kirchhoff's Voltage Law into the Faraday-henry-Lenz Law:

$$\oint_{\Gamma} \vec{E}(\vec{r}) \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

We can therefore get a definition of electromotive force (emf) around a closed loop:

$$emf = \frac{W}{q} = \oint_{\Gamma} \vec{E}(\vec{r}) \cdot d\vec{s}$$

This is the work done per unit charge by an electric field generated by a time-dependant magnetic flux. Therefore, electric fields generated by a variation in magnetic field flux are NOT conservative and can transfer energy to charges.

11.3 GENERATING EMF

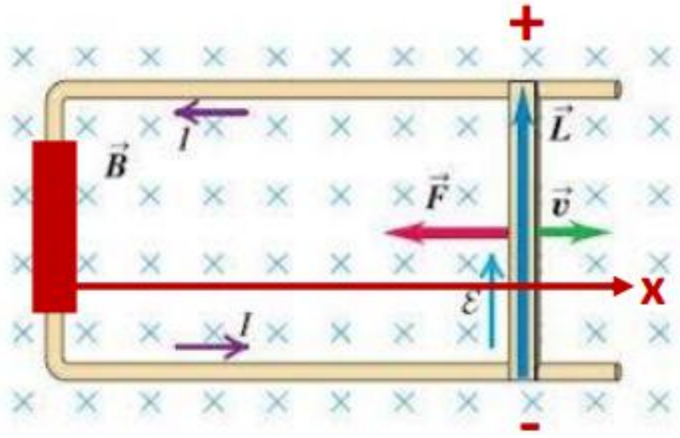
To generate emf in a circuit, there are three options:

- Vary the surface enclosed by the circuit (since flux depends on surface)
- Vary magnetic field over time
- Vary the angle between the field and the surface over time

If the magnetic field is uniform across the surface enclosed by the closed loop S, the surface is flat and the field is perpendicular to it, we get:

$$emf = -\frac{d\vec{B}(t)}{dt}S$$

In a rectangular circuit which is composed of a resistor (left), wires and a conductive bar (right) which can move on the wires, a magnetic field is applied (into the page)



Moving the bar of length L with velocity v will result in the surface area enclosed by the circuit to increase, therefore increasing flux. Hence:

$$emf = BvL$$
$$I = \frac{emf}{R} = \frac{BvL}{R}$$
$$F = ILB = \frac{B^2vL^2}{R}$$

The force (applied to the bar) is antiparallel to the movement.

To keep the bar moving, energy needs to be applied to the bar at a rate of

$$P = Fv = \frac{B^2v^2L^2}{R} = \frac{emf^2}{R}$$

Since

$$P_{electrical} = P_{mechanical}$$

Moving a metal bar through a uniform magnetostatic field proves that magnetic and electric fields are the same phenomenon observed from different relativistic points of view.

When the bar is moved across the magnetic field with velocity v, every charge feels a force

$$\vec{F} = q(\vec{v} \times \vec{B})$$

They therefore clump up at the ends of the bar (positive on one side and negative on the other).

HOWEVER, from the perspective of the bar, it is not moving relative to itself, and neither are the charges inside it. However, a force is still experienced on the charges, even with the frame of reference of the bar, therefore an electric field MUST be present, causing the force:

$$\vec{F} = q\vec{E}$$

We therefore get:

$$\vec{E} = \vec{v} \times \vec{B}$$

And the potential difference between the two sides of the bar (of length L):

$$\Delta V = EL = vBL = emf$$

We can also conclude that:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

IMPORTANT NOTE: The above equivalence ONLY applied when the potential difference/emf is generated by moving a conductor (in this case the bar) through a magnetic field.

EMF can be generated by rotating the surface. If a loop is rotated at constant angular velocity ω due to a torque τ:

$$\begin{aligned} emf &= BS\omega \sin \omega t \\ P &= \tau\omega \end{aligned}$$

Therefore, it is time-dependant and is AC.

11.4 MAXWELL EQUATIONS IN A VACUUM

LAW	INTEGRAL FORM	DIFFERENTIAL FORM
GAUSS FOR ELECTRIC FIELD	$\oint_S \vec{E}(\vec{r}) \cdot \vec{u}_N \, dS = \frac{Q}{\epsilon_0}$	$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
GAUSS FOR MAGNETIC FIELD	$\oint_S \vec{B}(\vec{r}) \cdot \vec{u}_N \, dS = 0$	$\nabla \cdot \vec{B} = 0$
FARADAY-HENRY-LENZ	$\oint_\Gamma \vec{E}(\vec{r}) \cdot d\vec{s} = -\frac{d}{dt}\left[\int_S \vec{B}(\vec{r}) \cdot \vec{u}_N \, dS\right]$	$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
AMPERE-MAXWELL	$\oint_\Gamma \vec{B}(\vec{r}) \cdot d\vec{s} = \mu_0 \, I + \mu_0 \, \epsilon_0 \frac{d\Phi(\vec{E})}{dt}$	$\nabla \wedge \vec{B} = \mu_0 \, \vec{j} + \mu_0 \, \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

12 INDUCTION AND AC CIRCUITS

If multiple magnetic fields superimpose:

$$\vec{B}_{TOT} = \Sigma\vec{B}_i$$
$$\Phi_{TOT} = \Sigma\Phi_i$$

We now take the case of two loops with currents I_1, I_2. Each will have a magnetic field of their own and will be affected by the other's.

Self-Inductance L describes how much a loop's own current creates flux through itself:

$$L = \frac{\Phi}{I}$$
$$[L] = H = WbA^{-1}$$

Where the unit is Henry.

Mutual Inductance M describes how one loop's current creates flix through another. In the example of loops 1 and 2,

$$M_{12} = M_{21} = \frac{\Phi_{21}}{I_1} = \frac{\Phi_{12}}{I_2}$$
$$[M] = H$$

Where Φ_{xy} is the flux observed on loop x due to loop y (format is $\Phi_{target,source}$). For two loops only, we can use M since the two mutual inductance values are identical.

We then have:

$$\Phi_1 = L_1 I_1 + M_{12} I_2$$
$$\Phi_2 = L_{2_{I_2}} + M_{21} I_1$$

L, M are defined by the geometrical parameters of the loops:

$$L = \int_{S_1} \vec{\xi}(\vec{r}) \cdot \vec{u}_N \, dS$$
$$M_{12} = \int_{S_2} \vec{\xi}_1(\vec{r}) \cdot \vec{u}_N \, dS$$

Recalling, in a solenoid with n loops per unit length, current I and loop surface area S:

$$B = \mu_0 n I$$

The flux through one loop is:

$$\Phi_1 = BS = \mu_0 n S I$$

And therefore, a piece of solenoid with $N = dn$ loops will have flux

$$\Phi_N = BSN = d\mu_0 n^2 S I$$

We then define flux per unit length

$$\Phi_{ul} = \frac{\Phi_N}{d}$$

to then get the inductance per unit length:

$$L = \frac{\Phi_{ul}}{I} = \mu_0 n^2 S$$
$$\Rightarrow L_{TOT} = Ld = d\mu_0 n^2 S$$
$$[L_{TOT}] = H, \qquad [L] = Hm^{-1}$$

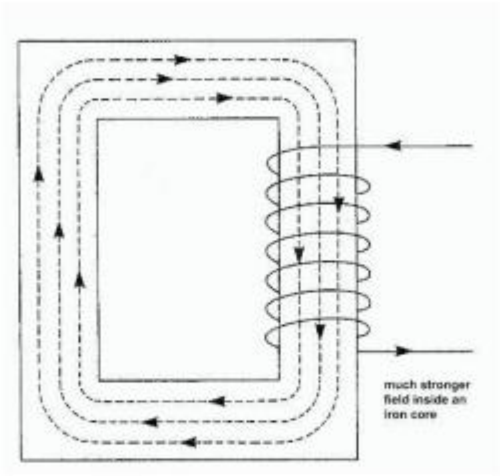
We can also define mutual inductance per unit length of two solenoids. Take two coaxial solenoids, first one with current I_1 going through it, n_1 loops per unit length, while the other has no current, has n_2 loops per unit length and loop area is A_2.

$$\Phi_2(B_1) = \mu_0 n_1 I_1 A_2$$
$$\Phi_{2,tot}(B_1) = \mu_0 n_1 I_1 N_2 A_2 = \mu_0 n_1 n_2 l A_2 I_1$$
$$\Phi_{2,ul}(B_1) = \frac{\Phi_{2,tot}(B_1)}{l} = \mu_0 n_1 n_2 A_2 I_1$$
$$M = \frac{\Phi_{2,ul}}{I_1} = \mu_0 n_1 n_2 A_2$$

If we wrap the solenoid around a ferromagnetic core (rod) we get a greater magnetic field, since:

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{\mathcal{M}}$$

This is how transformers work.



If we have self-inductance in a circuit:

$$\Phi = L_{TOT} I$$
$$emf = -\frac{d\Phi}{dt} = -\left(L_{TOT}\frac{dI}{dt} + I\frac{dL_{TOT}}{dt}\right)$$

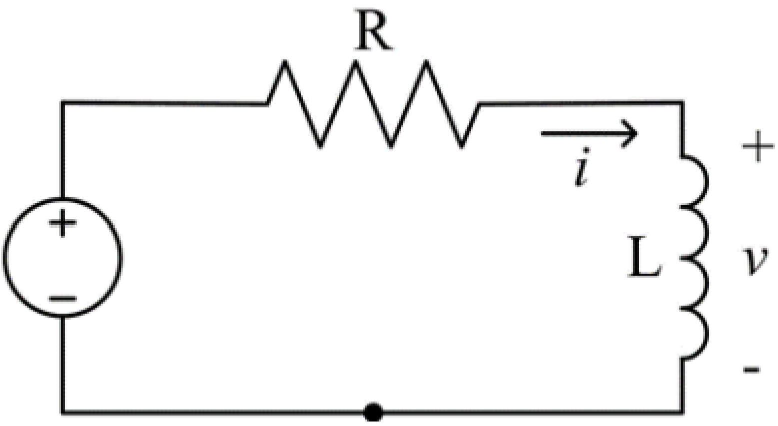
Since L_{TOT} (self-inductance) depends on length of the solenoid and on the shape, if the circuit is rigid, we have:

$$emf = -L_{TOT}\frac{dI}{dt}$$

An inductor is a solenoid in a circuit. They exploit self-inductance, and the following relationship holds:

$$V_L(t) = -emf(t) = L\frac{dI}{dt}$$

A good analogy for an inductor is an “electrical current flywheel” – it will resist a change in current. So, when a current is applied to a circuit with an inductor (diagram below, called an RL cirtuit), current will not instantly jump to the maximum $\frac{V}{R}$, but instead slowly increase, and the opposite happens when the current is removed.



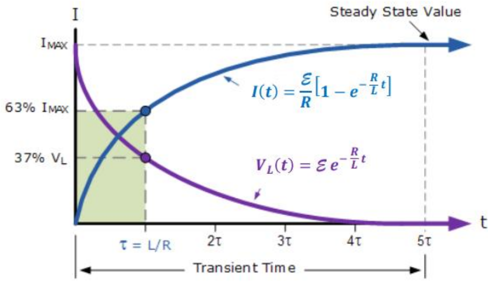
To know how I, V_L vary over time,

$$I(t) = \frac{\mathcal{E}}{\mathrm{R}}\left[1 - e^{-\frac{R}{L}t}\right]$$
$$V_L(t) = \mathcal{E}e^{-\frac{R}{L}t}$$

Where:

- $\mathcal{E}$ is the generator’s EMF (ideal voltage)
- R is the resistor’s resistance
- L is the inductor’s self-inductance
- t is the time after the current starts flowing into the circuit
- V_L is the voltage drop across the inductor

Which can be visualized as shown below:



τ is called the time constant of the circuit

$$\tau = \frac{L}{R}$$

When the circuit is shut off, the inverse happens:

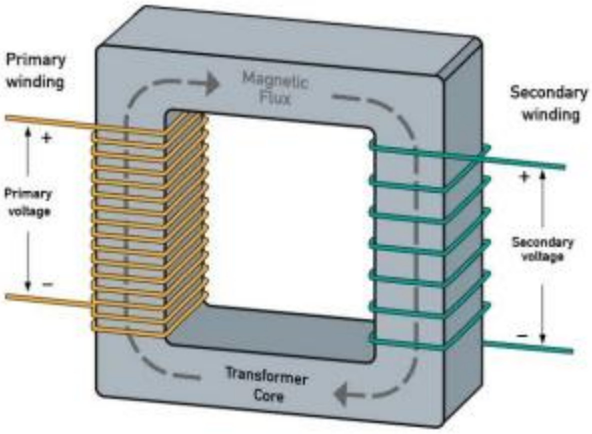
$$
\begin{aligned}
&I(t) = I_0 e^{-\tau t}\\
&|V_L(t)| = |V_R(t)| = RI_0 e^{-\tau t}\\
&P(t) = RI(t)^2\\
&\Rightarrow U = \frac{1}{2} LI_0^2 = \frac{1}{2\mu_0} B^2 \mathcal{V}
\end{aligned}
$$

Where U is the potential energy stored in the inductor and $\mathcal{V}$ is the volume inside the inductor, which is dissipated via heat through the resistor during the switch off. The right side of the final equation shows that energy is stored in the field itself rather than in the component. We then define energy density as:

$$
u_B = \frac{U}{\mathcal{V}} = \frac{B^2}{2\mu_0}
$$

12.1.1 Transformers

A transformer is used to change voltage and current **IN AN AC CIRCUIT**. It is shown below:



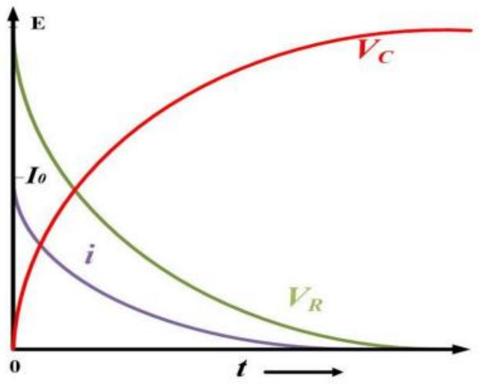
$$
\begin{aligned}
&\frac{P_1(t)}{P_2(t)} = 1\\
&\frac{I_2(t)}{I_1(t)} = \frac{emf_1(t)}{emf_2(t)} = \frac{N_1}{N_2}
\end{aligned}
$$

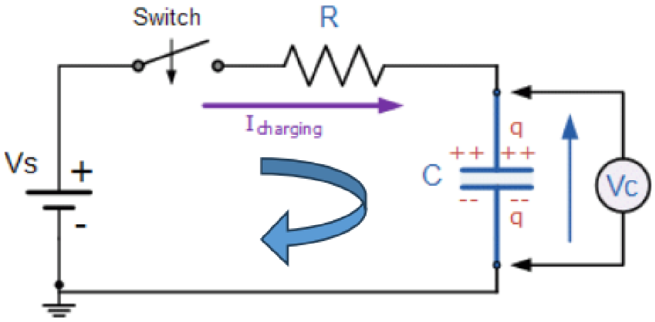
A TRANSFORMER DOES NOT WORK WITH DC CURRENT, SINCE NO MAGNETIC FIELD IS CREATED!

12.2 RC CIRCUITS

An RC circuit is one composed of a resistor of resistance R, a capacitor of capacitance C, a DC generator of voltage V_S and a switch. If the switch is turned on at $t = 0$, then

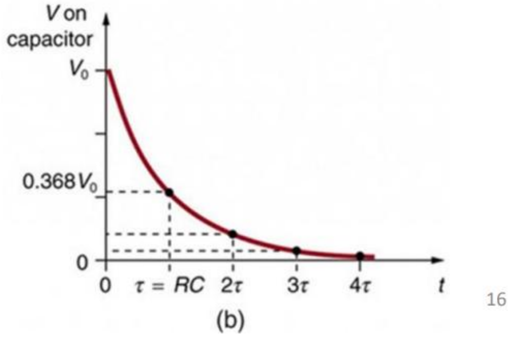
$$
\begin{aligned}
&I(t) = \frac{V_S}{R} e^{-\frac{1}{RC}t}\\
&V_C(t) = V_S - RI(t)\\
&q_C(t) = CV_S\left[1 - e^{-\frac{1}{RC}t}\right]\\
&\tau = RC
\end{aligned}
$$

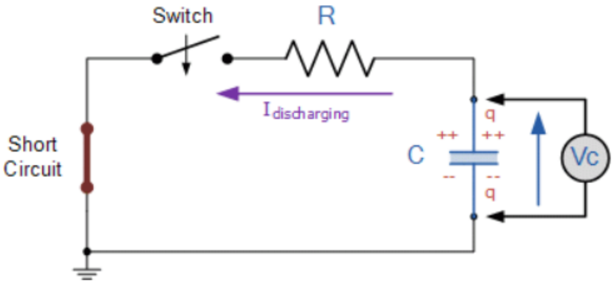




Now assume the generator is simply replaced with a wire (the capacitor is fully charge and starts discharging at $t = 0$):

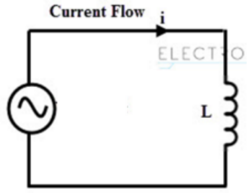
$$I(t) = \frac{V_{C0}}{R} e^{-\frac{1}{RC}t}$$
$$V_C(t) = RI(t)$$
$$q_C = CV_{C0}e^{-\frac{1}{RC}t}$$





12.3 AC CIRCUITS

Now take a circuit with an ideal AC Generator (With voltage $V_G(t)$) and an ideal inductor of inductance L:



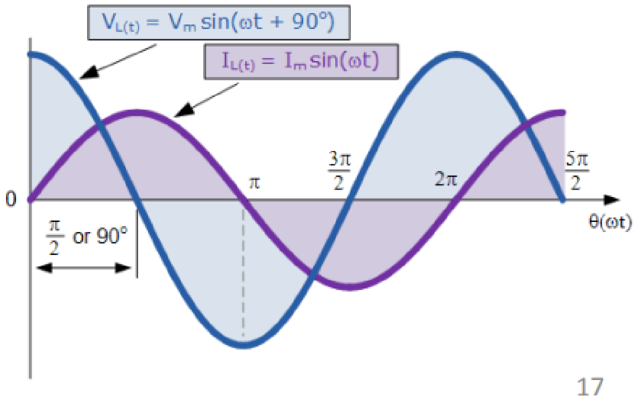
Where:

$$V_G(t) = V_m \sin\left(\omega t - \frac{\pi}{t}\right)$$

We then get:

$$V_L(t) = -V_G(t), \qquad V_{L,MAX} = -V_m$$
$$I(t) = \frac{V_{L,MAX}}{\omega L} \sin \omega t = I_{MAX} \sin \omega t$$
$$I_{MAX} = -\frac{V_m}{\omega L}$$

Therefore, current and voltage are out of phase by 90°. (Current has a $\frac{\pi}{2}$ phase delay)

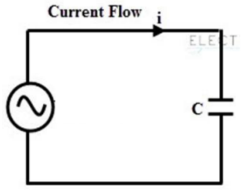


17

$$V_{L,MAX} = \omega L I_{MAX} = X_L I_{MAX}$$
$$X_L = \omega L$$

Where X_L is called inductive reactance, measured in ohms (Ω). It is zero when a DC current is present, and it increases when the frequency of the AC signal increases.

Now take an AC circuit with a capacitor of capacitance C:

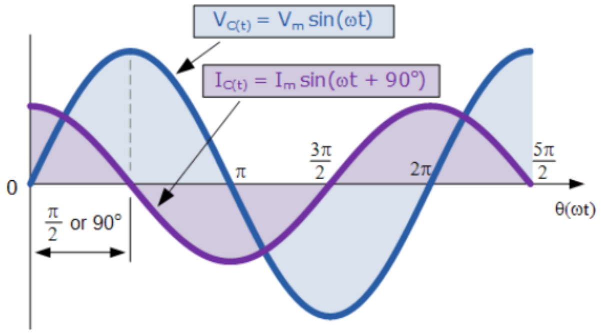


While having (note this time we have π rather than $\frac{\pi}{2}$):

$$V_G(t) = V_m \sin(\omega t + \pi)$$

We then have:

$$V_C(t) = V_{C,MAX} \sin \omega t\,, \qquad I(t) = I_{MAX} \sin\left(\omega t + \frac{\pi}{2}\right)$$
$$V_{C,MAX} = V_m = \left[\frac{1}{\omega C}\right] I_{MAX} = X_C I_{MAX}, \qquad I_{MAX} = \omega C V_m$$
$$X_C = \frac{1}{\omega C}$$



Where X_C is capacitive reactance, measured in Ω. It decreases when frequency of AC increases.

In this case, the current has a 90° phase **advance** and therefore means that the phase of the current is larger for inductors than for capacitors.

We can therefore conclude that inductors block high frequency signals, while capacitors block low frequency signals.

The analogy that can be used here is still the one of the heavy flywheels for inductors and a membrane in a pipe for capacitors: pushing through the membrane at low frequencies will lead to a very low amount of energy transmitted through; while vibrating it will make the membrane vibrate with you. As for the inductor, trying to spin back and forth a heavy flywheel many times a second will yield close to no result, while oscillating it slowly will work fine.

In an ideal system, no energy is consumed by an inductor and a capacitor in an AC circuit: all energy that went in during the positive power ($P = IV$) sections is given back when power is negative.

12.4 PHASORS AND IMPEDANCE

WARNING! COMPLEX NUMBERS AHEAD!

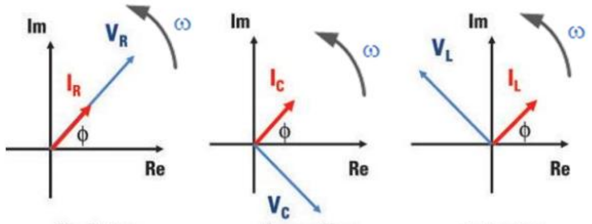
Consider AC circuits described as:

$$I(t) = I_{MAX} \sin \omega t = \mathrm{Im}\left[I_{MAX} e^{i\omega t}\right]$$
$$I(t) = I_{MAX} \cos \omega t = \mathrm{Re}\left[I_{MAX} e^{i\omega t}\right]$$

We can associate the current with a vector in the Gauss (complex) Plane having constant length I_{MAX}, forming an angle with the real axis which varies: $\theta = \omega t$. This vector is called a phasor.

We can easily associate a phasor to every electrical quantity of a circuit, such as:

- V, I across a resistor
- V, I across a capacitor
- V, I across an inductor



Resistor
Voltage in phase with current

Capacitor
Voltage lags current by 90°

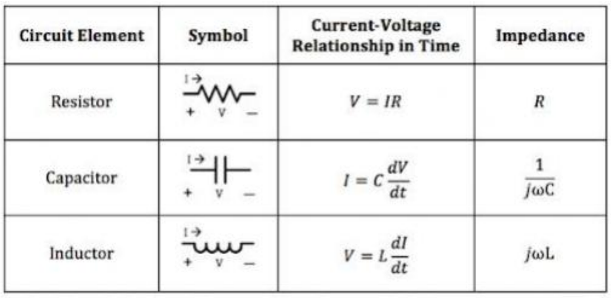
Inductor
Voltage leads current by 90°

In electrical engineering, variable notation can change to:

- $j = \sqrt{-1}$
- I, V being phasors and $|I|, |V|$ being their maximum values
- $i(t), v(t)$ being instantaneous current and voltage

We can define a value called impedance, it being a complex number which describes the relation between $\boldsymbol{V}, \boldsymbol{R}$ (phasors):

$$Z_R = R, \qquad Z_L = jX_L, \qquad Z_C = -jX_C$$



Circuit Element	Symbol	Current-Voltage Relationship in Time	Impedance
Resistor	I→ + V −	$V = IR$	R
Capacitor	I→ + V −	$I = C\frac{dV}{dt}$	$\frac{1}{j\omega C}$
Inductor	I→ + V −	$V = L\frac{dI}{dt}$	$j\omega L$

12.5 RLC CIRCUIT (SERIES)

The RLC Circuit is made up of a resistor (resistance R), a capacitor (capacitance C) and an inductor (inductance L) in series. In the first case, there will be no generator, we therefore have:

$$RI(t) + L\frac{dI}{dt} + \frac{1}{C}q_C(t) = 0, \qquad I(t) = \frac{dq_C}{dt}$$

Solving the differential equation for $I(t)$ by firstly differentiating the whole equation (left):

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I(t) = 0$$

Which is a homogeneous second-order ordinary differential equation, like the ones used for damped spring-mass systems.

$$I(t) = e^{-\frac{R}{2L}t}\left[Ae^{kt} + Be^{-kt}\right]$$

$$k = \omega_0\sqrt{\frac{R^2C}{4L} - 1}$$

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

Where A, B are constants to be found. (Given initial parameters for I_0, V_0). Since k is defined by a square root, it can have both real and complex solutions. Which lead to similar conclusions to the underdamped, overdamped and critically damped solutions for a damped harmonic oscillator.

- $R > 2\sqrt{\frac{L}{C}}$: Equivalent of overdamping. Current slowly decays without crossing 0. No oscillation.

$$I(t) = e^{-\frac{R}{2L}t}\left[Ae^{kt} + B^{-kt}\right]$$

- $R < 2\sqrt{\frac{L}{C}}$: Equivalent of underdamping. Current oscillates many times before slowly decaying.

$$I(t) = I_0e^{-\frac{R}{2L}t}\cos(\omega_{res}t + \phi)$$

$$\omega_{res} = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}$$

- $R = s\sqrt{\frac{L}{C}}$: Equivalent to critical damping.

$$I(t) = (1+t)Ae^{-\frac{R}{2L}t}$$

ω_0 is called the natural frequency. More on this later.

ω_{res} is the damped frequency. This is the corrected version of ω_0, accounting for R.

In an RLC circuit WITH a generator, we can't easily solve the differential equation, since it becomes non-homogeneous. We therefore use phasors:

$$\begin{aligned}&\boldsymbol{Z} = R + Xi = R + (X_L - X_C)i\\&|Z| = \sqrt{R^2 + (X_L - X_C)^2}\\&\tan\theta_Z = \frac{X_L - X_C}{R}\end{aligned}$$

$$\begin{aligned}&\therefore \text{ if } X_L = X_C \Rightarrow \vec{V}_L + \vec{V}_C = 0\\&\Rightarrow X = 0\\&\Rightarrow \boldsymbol{Z} = R\end{aligned}$$

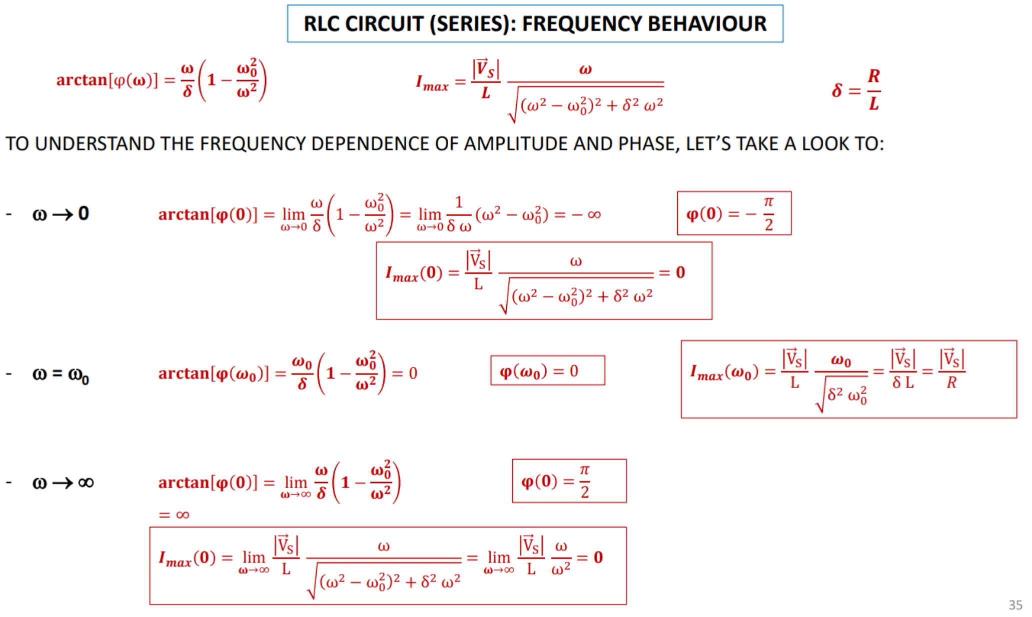
For fixed components, X will depend on frequency, since:

$$X = \omega L - \frac{1}{\omega C}$$

When $X = 0$, the circuit is at its resonant frequency. Here, voltage and current are in phase, since reactance is zero, and therefore impedance is minimized. This frequency is f_0, called the resonant frequency (Measured in hertz, while ω_0 is the same thing, just measured in radians per second).

$$\begin{aligned}\omega_0 &= \frac{1}{\sqrt{LC}}\\ f_0 &= \frac{1}{2\pi\sqrt{LC}}\end{aligned}$$

This shows whatever the fuck this is:



RLC CIRCUIT (SERIES): FREQUENCY BEHAVIOUR

$$\arctan[\varphi(\omega)] = \frac{\omega}{\delta}\left(1 - \frac{\omega_0^2}{\omega^2}\right) \qquad I_{max} = \frac{|\vec{V}_S|}{L}\frac{\omega}{\sqrt{(\omega^2-\omega_0^2)^2 + \delta^2\,\omega^2}} \qquad \delta = \frac{R}{L}$$

TO UNDERSTAND THE FREQUENCY DEPENDENCE OF AMPLITUDE AND PHASE, LET'S TAKE A LOOK TO:

- $\omega \rightarrow 0$ $\quad \arctan[\varphi(0)] = \lim_{\omega\to 0}\frac{\omega}{\delta}\left(1 - \frac{\omega_0^2}{\omega^2}\right) = \lim_{\omega\to 0}\frac{1}{\delta\,\omega}(\omega^2 - \omega_0^2) = -\infty$ $\quad \boxed{\varphi(0) = -\frac{\pi}{2}}$

  $\boxed{I_{max}(0) = \frac{|\vec{V}_S|}{L}\frac{\omega}{\sqrt{(\omega^2-\omega_0^2)^2 + \delta^2\,\omega^2}} = 0}$

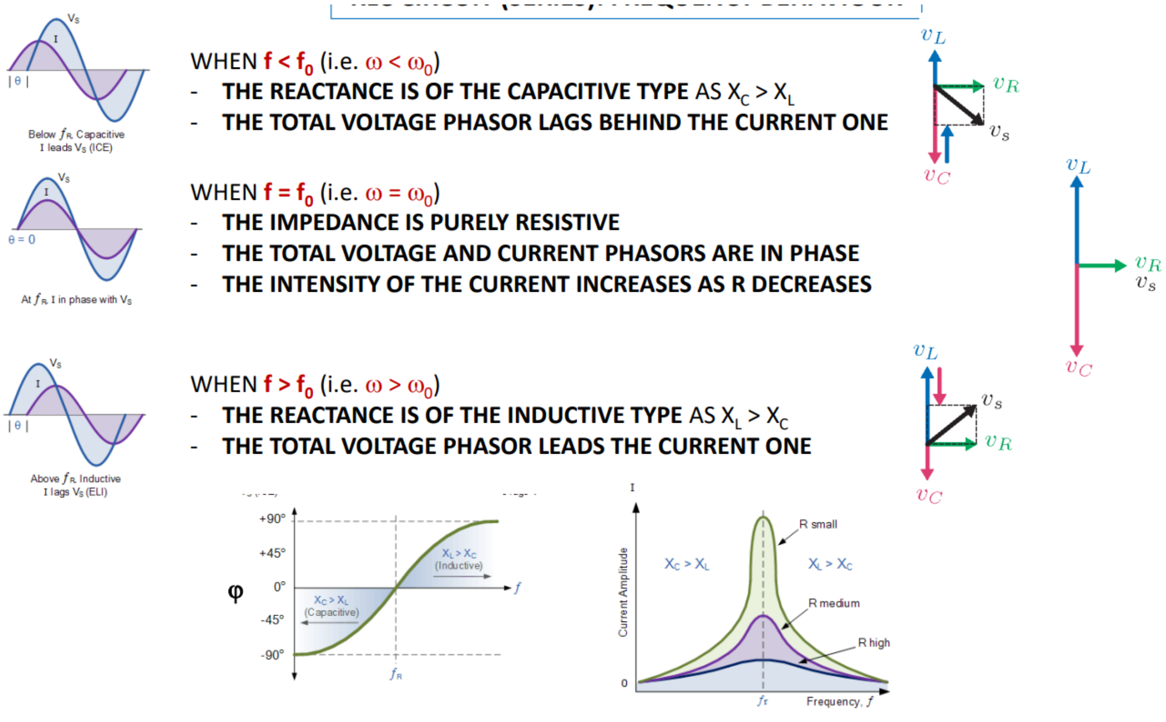
- $\omega = \omega_0$ $\quad \arctan[\varphi(\omega_0)] = \frac{\omega_0}{\delta}\left(1 - \frac{\omega_0^2}{\omega^2}\right) = 0$ $\quad \boxed{\varphi(\omega_0) = 0}$ $\quad \boxed{I_{max}(\omega_0) = \frac{|\vec{V}_S|}{L}\frac{\omega_0}{\sqrt{\delta^2\,\omega_0^2}} = \frac{|\vec{V}_S|}{\delta\,L} = \frac{|\vec{V}_S|}{R}}$

- $\omega \rightarrow \infty$ $\quad \arctan[\varphi(0)] = \lim_{\omega\to\infty}\frac{\omega}{\delta}\left(1 - \frac{\omega_0^2}{\omega^2}\right) = \infty$ $\quad \boxed{\varphi(0) = \frac{\pi}{2}}$

  $\boxed{I_{max}(0) = \lim_{\omega\to\infty}\frac{|\vec{V}_S|}{L}\frac{\omega}{\sqrt{(\omega^2-\omega_0^2)^2 + \delta^2\,\omega^2}} = \lim_{\omega\to\infty}\frac{|\vec{V}_S|}{L}\frac{\omega}{\omega^2} = 0}$

35

Based on frequency compared to resonance frequency, we have different types of impedance:



WHEN **f < f$_0$** (i.e. $\omega < \omega_0$)
- **THE REACTANCE IS OF THE CAPACITIVE TYPE** AS $X_C > X_L$
- **THE TOTAL VOLTAGE PHASOR LAGS BEHIND THE CURRENT ONE**

WHEN **f = f$_0$** (i.e. $\omega = \omega_0$)
- **THE IMPEDANCE IS PURELY RESISTIVE**
- **THE TOTAL VOLTAGE AND CURRENT PHASORS ARE IN PHASE**
- **THE INTENSITY OF THE CURRENT INCREASES AS R DECREASES**

WHEN **f > f$_0$** (i.e. $\omega > \omega_0$)
- **THE REACTANCE IS OF THE INDUCTIVE TYPE** AS $X_L > X_C$
- **THE TOTAL VOLTAGE PHASOR LEADS THE CURRENT ONE**

Note:

$$\omega_{res} = \omega_0 \sqrt{1 - \frac{R^2C}{4L}}$$

This meaning that the resonance frequency isn't actually ω_0, but we would need to adjust for resistance. If R is high, there will be heavy damping of the frequency (The curve flattens and widens), and the inverse for small resistance (The curve becomes sharper around the peak). This is referred to as the Q-factor, where high Q means narrower and sharper curves.

$$q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

We then can get the formula for power:

$$\bar{P} = \frac{U_{cycle}}{T} = \frac{V_S I_{MAX}}{2} \cos\varphi$$

Where:

- U_{cycle} is the energy consumed per cycle
- T is the length of the cycle in seconds
- V_S is the maximum voltage at the source
- I_{MAX} is the peak current
- φ is the phase difference between current and voltage

Therefore, if the phase difference is close to $\frac{\pi}{2}$, the power needed is low, since the capacitors and inductors store most of the energy (rather than it being released as heat by the resistor).

Another form of the power formula is one using root mean square (rms):

$$\bar{P} = V_{rms}I_{rms}\cos\varphi = RI_{rms}^2\cos\varphi = \frac{V_{rms}^2\cos\varphi}{\sqrt{(X_L - X_C)^2 + R^2}}$$

Where rms is calculated using

$$x_{rms} = \sqrt{\frac{1}{T}\int_0^T [x(t)]^2 dt}$$

Which in discrete terms, is the square root of the average between the squares of all terms.

13 WAVES

A wave is a physical perturbation that carries energy, but no matter. It is periodic in time and space. Generally, in 1D:

$$f(x,t) = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$
$$k = \frac{2\pi}{\lambda}$$
$$\nu = \frac{1}{T}$$
$$\omega = 2\pi\nu = \frac{2\pi}{T}$$

Where:

- λ is the wavelength
- k is the wavenumber
- T is the period
- ν is the frequency (note: this is not the letter v, but is the Greek letter “nu”)
- ω is the angular frequency

Not all waves are sinusoidal, but Fourier Theorem states that any periodic function can be produced by the superposition of an appropriate number of sine and cosine waves of suitable frequency, wavelength and amplitude.

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)$$
$$a_0 = \frac{1}{L}\int_0^L f(x)dx$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

To describe a wave, a function f of space and time must satisfy the wave equation:

$$\nabla^2 f = \frac{1}{v^2}\frac{\delta^2 f}{\delta t^2}$$

And

$$v = \lambda\nu = \omega k$$

For travelling waves, we have:

$$\varphi = \frac{2\pi}{\lambda}(x - vt) = k(x - vt)$$
$$\frac{\delta\varphi}{\delta t} = -kv = \omega$$
$$v_\varphi = \frac{\omega}{k}$$

$$\frac{\delta\varphi}{\delta x} = k$$

Here, φ is the phase, and v_φ is called the phase velocity.

To describe a wave that is localized in space (a “packet”) we use the Fourier Integral, rather than Fourier Transform:

$$f(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(k) e^{i[kx-\omega(k)t]} dk$$

Here, $A(k)$ indicates the weight of a specific sinusoidal wave. We can take Δk as being the width (range) of the region where the waves are relevant. We then have Δx being the width of the packet in space.

Then it must be true that:

$$\Delta x \Delta k \geq 1$$

Since the tighter the packet, the more waves will be needed to describe it.

A similar relationship can be observed between Δt (time during which the wave exists) and $\Delta\nu$ (the number of frequencies needed to describe the packet):

$$\Delta\nu\Delta t \geq 1$$

For a sine wave, $\Delta\nu = 0 \Rightarrow \Delta t = \infty$ since it never ends.

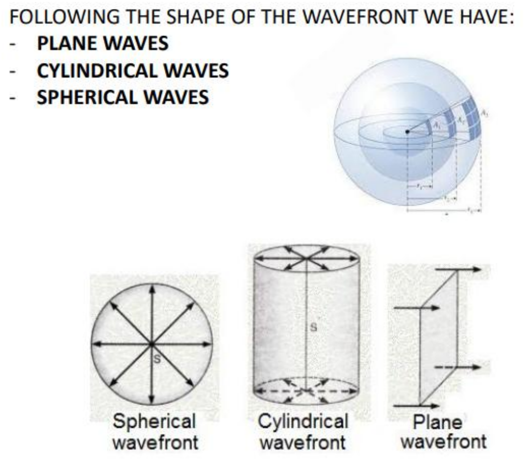
Group velocity is the speed at which information carried by the packets travel at:

$$v_g = \frac{\delta\omega}{\delta k}$$

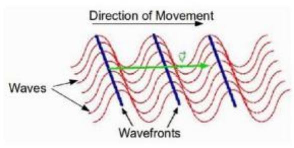
If all components have the same v_φ, the shape of the wavepacket does not change over time. This wave is called non-dispersive. If v_φ is different for each frequency, then some components will travel faster than other, and this wave is called dispersive.

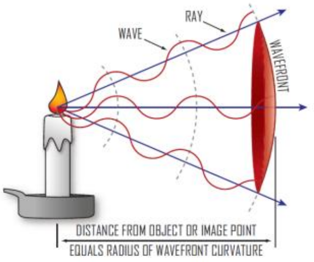
EM waves have transverse polarization, since oscillations are perpendicular to direction. Sound waves have longitudinal polarization since the oscillations are in the direction of travel.

A wavefront is a neighbourhood of a wave which have the same phase:



FOLLOWING THE SHAPE OF THE WAVEFRONT WE HAVE:
- **PLANE WAVES**
- **CYLINDRICAL WAVES**
- **SPHERICAL WAVES**





13.1 STANDING WAVES

Standing waves occupy their whole existence domain (the space in which they exist). They will only vary their shape in time.

We now consider a string of length L which is fixed on both ends

$$\Rightarrow A(0,t) = A(L,t) = 0$$

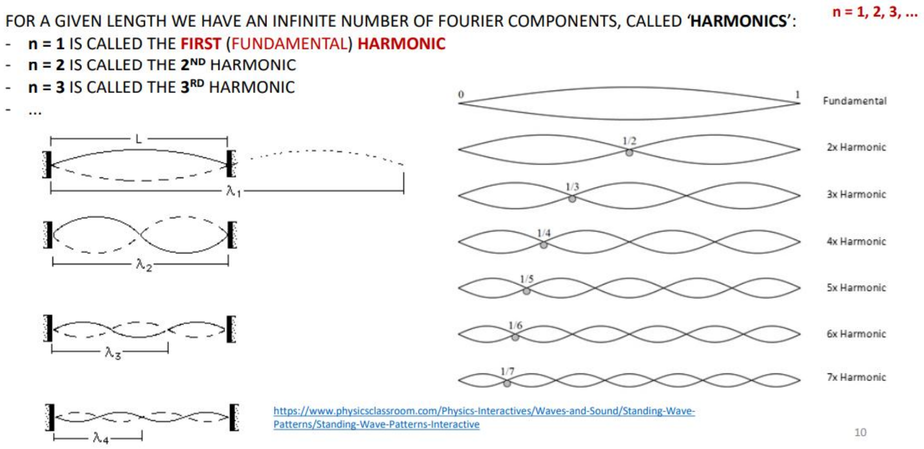
We then can derive that:

$$A_n(x,t) = A_{MAX,n}\sin(k_n x)\cos(\omega_n t)$$

$$\omega_{\mathrm{n}} = v_\varphi \frac{2\pi}{\lambda_n}$$

$$\nu_n = \frac{v_\varphi}{2L} = n\frac{v_\varphi}{2L} \Rightarrow \nu_n = n\nu_1$$

$$n = 1,2,3,\ldots$$



FOR A GIVEN LENGTH WE HAVE AN INFINITE NUMBER OF FOURIER COMPONENTS, CALLED '**HARMONICS**':

n = 1, 2, 3, ...

- **n = 1** IS CALLED THE **FIRST** (FUNDAMENTAL) **HARMONIC**
- **n = 2** IS CALLED THE 2^{ND} HARMONIC
- **n = 3** IS CALLED THE 3^{RD} HARMONIC
- ...

https://www.physicsclassroom.com/Physics-Interactives/Waves-and-Sound/Standing-Wave-Patterns/Standing-Wave-Patterns-Interactive

Points where amplitude is always zero are called nodes. There are $n-1$ nodes for each harmonic. (The extremes don't count)

We can deduce that harmonics are evenly spaced in frequency, and frequency of harmonics depends on phase frequency.

Phase velocity in a string is given by:

$$v_\varphi = \sqrt{\frac{T}{\mu}}$$

Where T is the tension in the string and μ is the linear mass density (mass per unit length) measured in $kg\ m^{-1}$. Therefore, we can get the frequency of the n-th harmonic:

$$\nu_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

When we say that a string is vibrating at 200Hz, it means that $\nu_1 = 200Hz$, however there can be other harmonics in it, with lower weight.

13.2 ELECTROMAGNETIC WAVES

$$v_{em} = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299{,}792{,}458\ ms^{-1} \approx 3 \times 10^8$$

An accelerating charge creates an electromagnetic field. It will propagate EM waves in all directions except that of acceleration, and of a specific frequency if it has oscillating motion.

In an EM wave **IN A VACUUM**:

- The electric field and the magnetic field are perpendicular
- A well-defined relationship is defined between their intensities
- They are perpendicular to direction of propagation
- The direction of propagation is defined so that

  $$\vec{k} \times \vec{E} = \omega \vec{B}$$
  $$|\vec{E}| = c|\vec{B}|$$

  Where:

  $$\vec{k} = k u_k$$
  $$k = \frac{2\pi}{\lambda}$$

  And u_k is the unit direction indicating the direction of the wave.

When an EM waves travels, it carries energy. We can define the “Poynting Vector” as:

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

We can simplify this to:

$$|\vec{S}| = \frac{1}{A}\frac{dU}{dt} = c\varepsilon_0 |\vec{E}|^2$$

14 LIGHT AND OPTICAL PROPERTIES

Light is an EM wave. It has Intensity I (measured in $W\ sr^{-1}$) and an irradiance E (measured in Wm^{-1}, which is power per unit area).

When we send a light beam of intensity I_0 on a thick surface, part I_R is reflected, part I_T is transmitted through, and part I_A is absorbed. Energy is conserved so:

$$I_0 = I_T + I_R + I_A$$
$$1 = \frac{I_T}{I_0} + \frac{I_R}{I_0} + \frac{I_A}{I_0} = T + R + A$$

Where:

- T is the coefficient of transmission
- R is the coefficient of reflection
- A is absorbance

These coefficients depend on the optical properties of the medium itself, the medium from which the light comes from and the medium to which the light will be transmitted to.

These properties are:

- Speed of light in it
- A parameter that describes the amount of absorption
  - Dielectric materials do not absorb light
  - Conductors absorb a fraction of light intensity per unit length

A material with dielectric constant $\varepsilon = \varepsilon_0\varepsilon_r$ and magnetic susceptibility $\mu = \mu_0\mu_r$ will have speed of light in it of:

$$v = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{\varepsilon_r\mu_r}}c$$

In non-ferromagnetic materials (dielectrics usually belong here)

$$\mu_r \approx 1$$
$$\Rightarrow v = \frac{1}{\sqrt{\varepsilon_r}}c = \frac{c}{n}$$

Where n is the refractive index (or index of refraction/IOR), which is always > 1 in dielectrics.

For a wave with wavelength λ_0 in a vacuum, its wavelength in a medium IOR n,

$$v = \lambda f = \frac{c}{n}$$
$$\Rightarrow \lambda = \frac{v}{f} = \frac{c}{nf} = \frac{\lambda_0}{n}$$
$$k = nk$$

Absorption properties of a material are given by either the absorption coefficient α or the extinction coefficient κ.

Beer-Lambert Law states that if a light beam with an intensity I_0 enters a medium with absorption coefficient α, the intensity of the beam after travelling a distance x inside the medium is:

$$I(x) = I_0 e^{-\alpha x}$$

We can also say that:

$$\alpha = \frac{4\pi\kappa}{\lambda_0}$$
$$[\alpha] = m^{-1}$$

With this new value κ, we can introduce the "Complex Refractive Index":

$$\mathcal{N} = n + i\kappa$$

Which we can use in:

$$E(x,t) = E_0 e^{i[\mathcal{N} k_0 x - \omega t]}$$

$$I \propto E^2$$

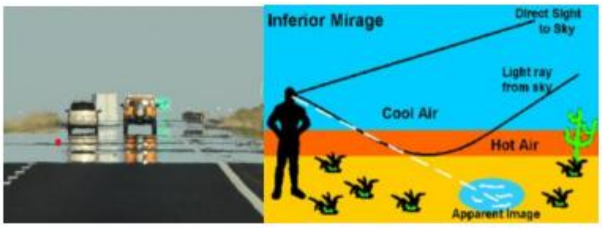
14.1 FERMAT'S PRINCIPLE

Fermat's Principle states that "To move from one point to another, light follows the path requiring the least time (or path of least action in mechanics)."

Sometimes, the shortest path is not the quickest.

Mirages and "Fata Morgana Effect" are caused by:

- Density of dry air increases when temperature increases
- Refractive index of air when density increases

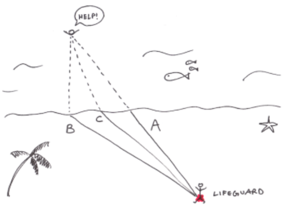


If θ_i is the angle of incidence and θ_r is the angle of reflection, Fermat's Principle will lead to

$$\theta_i = \theta_r$$

This law is true for flat surfaces (specular reflection). For rough surfaces, light is scattered in all directions (diffused reflection).

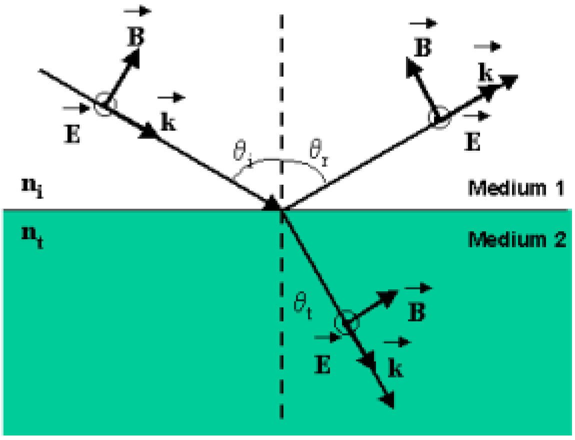
Holy hell, V-Sauce reference:

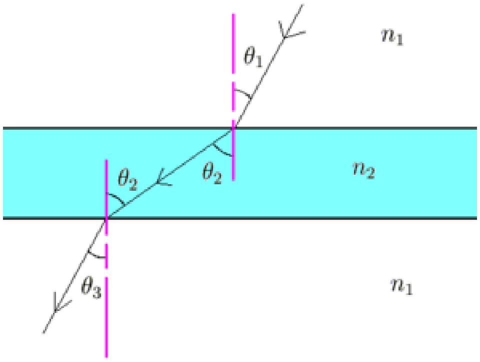


When a light ray is directed from one non-absorbing medium to another, we usually have a transmitted ray and a reflected ray. All rays lay in the same plane, called the plane if incidence.

We can get the angle of refraction (all angles here are with respect to the vertical axis) θ_t, we have:

$$n_i \sin\theta_i = n_t \sin\theta_t$$





Since a flat surface cannot absorb (it has no thickness), we have:

$$T + R = 1 \Rightarrow T = 1 - R$$

$$R = \frac{(n_2 - n_1)^2 + (\kappa_2 - \kappa_1)^2}{(n_2 + n_1)^2 + (\kappa_2 + \kappa_1)^2}$$

For air, $n \approx 1, \kappa \approx 0$.

For light going from air to metals, the formula can be greatly simplified via approximations,

$$n \ll \kappa$$
$$\Rightarrow R \approx 1$$

Which is why metals are very reflective. A high value for κ means that most of the light is absorbed immediately, and this energy will cause vibrations in electrons, which will therefore instantly emit the light back.

If we consider the second medium to be non-absorbing and the medium of origin being air, we have:

$$R = \frac{(n_2 - 1)^2}{(n_2 + 1)^2}$$
$$T = \frac{4n_2}{(n_2 + 1)^2}$$

Generally, for glass, $R = 0.04$, so only about 4% of light is reflected per surface (since a window has a surface on each side, light is reflected internally as well, leading to about 7.5% of light to be reflected. The rest is transmitted through.

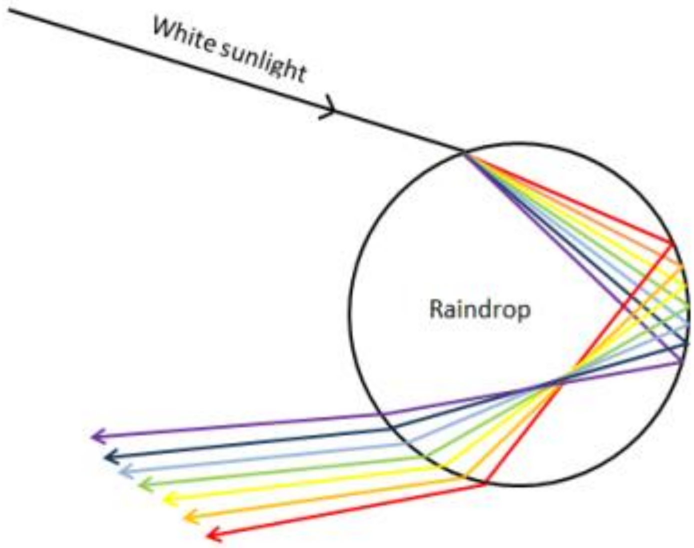
Transmission coefficient varies with wavelength, which is why we have coloured glass.

Total reflection happens when the angle of incidence θ_i is greater than the critical angle of incidence $\theta_{i,cr} < 90°$.

$$n_i \sin \theta_{i,cr} = n_t$$
$$\Rightarrow \theta_{i,cr} = \arcsin \frac{n_t}{n_i}$$

This is how optic fibres work.

Refractive index of water increases as wavelength decreases, which is why rainbows exist:



15 BEATINGS, INTERFERENCE AND DIFFRACTION

If two waves of the same nature meet at a given point, the effect perceived is the combination of the two. This is called waves superposition.

To do this, we sum the amplitudes of the two waves. We can also do it using phasors.

$$A_{tot}(x,t) = A_{01}\sin(k_1x-\omega_1t) + A_{02}\sin(k_2x-\omega_2t)$$
$$\vec{A}_{tot}(x,t) = A_{01}e^{i(k_1x-\omega_1t)} + A_{02}e^{i(k_2x-\omega_2t)}$$

When the frequencies of the two waves are very close, the superposition phenomenon is called "beatings", while if they are exactly equal, it is called interference.

15.1 BEATINGS

Since the two frequencies are almost equal, we can write (And for simplicity, $A_{01} = A_{02} = A_0$):

$$k = \frac{k_1+k_2}{2}, \frac{\Delta k}{2} = \frac{k_1-k_2}{2} \ll k$$
$$\omega = \frac{\omega_1+\omega_2}{2}, \frac{\Delta\omega}{2} = \frac{\omega_1-\omega_2}{2} \ll \omega$$
$$A_{tot}(x,t) = \left\{A_0\cos\left[\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right]\right\}\sin[kx-\omega t]$$

The high frequency component is called the carrier, and the low frequency component is called the modulator since it varies the amplitude of the carrier.

The shape of the modulator is the envelope of the signal.

Since a monochromatic wave cannot carry information, its amplitude can be modulated over time. I these frequencies are in the range of sound wave frequencies we deal with AM (amplitude modulation) broadcasting.

With AM broadcasting, the amplitude of the carrier is modulated by sound data, which is then transmitted to the receiver. The signal is then filtered from the carrier signal, remaining with the modulating signal, which is then sent to speakers.

15.2 INTERFERENCE

When two waves of the same type and frequency meet at point P, we have interference:

$$A_{tot}(P,t) = A_{01}\sin\varphi_1(P,t) + A_{02}\sin\varphi_2(P,t)$$

If we define Δs_1 as the distance from point P to the origin of wave 1, and Δs_2 as the distance from point P to the origin of wave 2, we can get the phase difference (Time independent since both have the same frequency)

$$\Delta\varphi(P) = \Delta\varphi_0 + k[\Delta s_2 - \Delta s_1] = \Delta\varphi_0 + \frac{2\pi}{\lambda}[\Delta s_2 - \Delta s_1]$$

Where $\Delta\varphi_0$ is the phase difference at the origin of the sources. Peak intensity depends on phase difference. If $\Delta\varphi = 0$, then they constructively interfere, leading to the sum of the two peak intensities. If $\Delta\varphi = 180°$, then we have destructive interference and the two waves always cancel out.

If $A_{tot}(P)$ is the combined peak intensity at point P,

$$A_{tot}(P) = \sqrt{A_{01}^2 + A_{02}^2 + 2A_{01}A_{02}\cos\Delta\varphi}$$

(Cosine rule). If $A_{01} = A_{02} = A_0$,

$$A_{tot}(P) = A_0\sqrt{2(1 + \cos\Delta\varphi)}$$

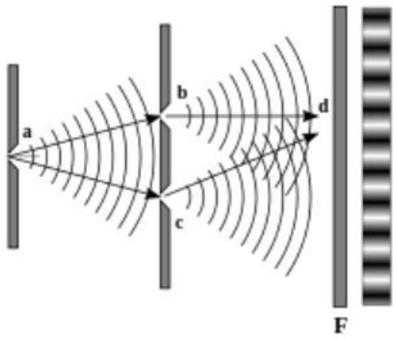
We also have:

$$I(P,t) = \eta\, A(P,t)^2$$

Where $\eta > 0$ is specific to each type of wave.

Wavepackets only last about $10^{-8}s$, and each packet has a random phase. Therefore, we cannot catch interference with normal light bulbs: we only capture the average intensity over longer periods of time than the duration of each interference pattern. Lasers are the solution to this. Their wavepackets can last for seconds or minutes, and their interference patterns are visible.

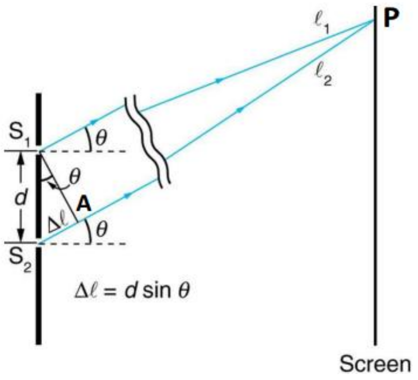
Young's Interferometer:



The interference pattern in Young's Interferometer is constant in time.

We can make a long-distance approximation for phase difference:

$$\Delta\varphi(P) = \frac{2\pi}{\lambda} d\sin\theta$$



We can then find the intensity minima and maxima:

$$\sin\theta_{MIN} = \left(m + \frac{1}{2}\right)\frac{\lambda}{d}, \qquad m \in \mathbb{Z} : \left|\left(m + \frac{1}{2}\right)\frac{\lambda}{d}\right| \leq 1$$
$$\sin\theta_{MAX} = m\frac{\lambda}{d}, \qquad m \in \mathbb{Z}: m \in \mathbb{Z} : \left|m\frac{\lambda}{d}\right| \leq 1$$

The minima and maxima will be uniformly spaced OVER θ. The pattern is symmetrical.

If $d < \lambda$, only the central maximum is present.

Since interferences usually occur with waves that are not the exact same amplitude, absolute darkness will not usually occur, since peak intensity at minima will not be zero.

$$I(P,t) = I_{MAX}(P)\sin^2\omega t$$

A coherent source is one that emits waves having the same wavelength (and frequency) and emits them with a consistent phase difference in time. These sources can have a direction due to interference patterns. To improve directionality (make bright areas brighter and smaller) we can add more coherent sources. A coherent source can be a slit in Young's Interferometer.

15.3 N-SLIT INTERFERENCE

When we have interference patterns between N coherent sources:

$$I(\theta) = I_0\left[\frac{\sin N\alpha}{\sin\alpha}\right]^2$$
$$\alpha = \frac{\pi d\sin\theta}{\lambda}$$

This tells us that:

$$I_{MAX} \propto N^2$$
$$\ell \propto \frac{1}{N}$$

Where ℓ is the width of each peak.

In this scenario, between main peaks, smaller secondary peaks are present in between. The intensity of the secondary peaks is equal to I_0. The primary maxima have half width of $\frac{2\pi}{N}$. (half width starts at a maximum and ends at the first minimum)

Between each two primary maxima, there are $N - 2$ secondary maxima and $N - 1$ minima.

If the distance between two coherent sources is lower than the wavelength emitted, then there will only be one central maximum.

By changing the phase difference at emission between sources, we can change the direction of the peaks (or single peak if $d < \lambda$) and therefore send a directional signal in space without rotating the device physically.

15.4 DIFFRACTION

Diffraction occurs when a wavefront goes through a hole or when a part of the wavefront is removed from the propagating wave.

At short distances, Fresnel Diffraction is the phenomenon of interest, while at long distances Fraunhofer Diffraction is used.

When light squeezes through a small hole (slit width a), it spreads out. The smaller the hole, the wider the spread. Each point in the whole acts as an individual coherent source, meaning that the slit has infinite coherent sources. We assume each source becomes a source of spherical waves. For simplicity, we divide it into $N = \frac{a}{\Delta y}$ sources.

The central maximum will have amplitude and intensity of:

$$A_C = NA_0$$
$$I_C = N^2 I_0$$

We can then conclude that:

$$I(\theta) = I_0 \left[\frac{\sin \alpha}{\alpha}\right]^2 = I_0 \operatorname{sinc}^2 \alpha$$
$$\alpha = \frac{\pi a \sin \theta}{\lambda}$$
$$\operatorname{sinc} x = \frac{\sin x}{x}$$

Then,

$$\sin \theta_{MIN} = n\frac{\lambda}{a}, \qquad n \in \mathbb{Z} \setminus \{0\}$$

WARNING! $n = 0$ is the central maximum, not a minimum.

The width of the central maximum is $a\frac{\lambda}{a}$ (trough-to-trough)

Since amplitude of maxima varies monotonically with θ, the following formula for maxima is not exact, but is a good-enough approximation:

$$\sin \theta_{MAX} \approx \left(m + \frac{1}{2}\right)\frac{\lambda}{a}$$

If $a < \lambda$, light spreads everywhere, since the angle for the first minimum has no real solutions. We can assume that the intensity is near uniform (this is not fully correct though).