

## Inverse Functions

- $f$  surjective onto  $B$  if
  - $\forall y \in B, \exists x \in \text{dom}f: y = f(x)$
- $f$  injective in  $A$  if
  - $\forall x_1, x_2 \in A: x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
  - $\forall x_1, x_2 \in A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- $f$  bijective (or one-to-one) if  $f(A) = B$  and  $f$  is injective in  $A$

## Function Properties

- Maximum  $\neq$  max point
  - Max is unique, max point is not. (e.g. max for  $y = \sin x$  is 1, but max points are  $\{(x, 1): \frac{x}{2\pi} \in \mathbb{Z}\}$ )
- $f(x) = x^2$  is convex
- $f(x) = -x^2$  is concave

## Definition of Limits

$$\lim_{x \rightarrow \gamma} f(x) = \lambda, \gamma, \lambda \in \mathbb{R}^*$$

means that

$$\forall I(\lambda) \exists I(\gamma)$$

$$\forall x: x \in \text{dom}f \cap I(\gamma) \setminus \{\gamma\} \Rightarrow f(x) \in I(\lambda)$$

## Local Boundedness Theorem

$$\lim_{x \rightarrow \gamma} f(x) = l \in \mathbb{R}$$

$$\Rightarrow \exists I(\gamma): f(x) \text{ is bounded in } I(\gamma) \setminus \{\gamma\}$$

## Sign and Limit Theorem

If sign of limit is +ve, then there exists a neighbourhood of the point where the function is positive. Same with negatives

If the limit at the point is defined, then the opposite can be said: if interval +ve, then lim +ve.

## I Comparison Theorem

$f, g$  defined in  $I(\gamma) \setminus \{\gamma\}$

$$\lim_{x \rightarrow \gamma} f(x) = \lambda, \lim_{x \rightarrow \gamma} g(x) = \mu$$

$$\exists I(\gamma): f(x) \leq g(x)$$

$$\Rightarrow \lambda \leq \mu$$

## II Comparison Theorem

$f, g, h$  defined in  $I(\gamma) \setminus \{\gamma\}$

$$f(x) \leq g(x) \leq h(x)$$

$$\lim_{x \rightarrow \gamma} f(x) = \lim_{x \rightarrow \gamma} h(x) = l \in \mathbb{R}$$

$$\Rightarrow \lim_{x \rightarrow \gamma} g(x) = l$$

## Infinite Case

$f, g$  defined in  $I(\gamma) \setminus \{\gamma\}$

$$f(x) \leq g(x)$$

$$\lim_{x \rightarrow \gamma} f(x) = +\infty$$

$$\Rightarrow \lim_{x \rightarrow \gamma} g(x) = +\infty$$

## Differentiation

$$\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{g(x)^2}$$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

A cusp point is a point where L derivative is  $\pm\infty$  but R derivative is  $\mp\infty$

## Intermediate Value Theorem- I Version (Weak Version)

$f$  continuous on  $[a, b]$

$$\Rightarrow f([a, b]) \supseteq [f(a), f(b)]$$

## Weierstrass Theorem

$f$  continuous on  $[a, b]$

1.  $\Rightarrow f$  bounded in  $[a, b]$
2.  $\Rightarrow f$  admits the absolute maximum  $M$  and absolute minimum  $m$  in  $[a, b]$ 
  - a.  $\exists x_M: f(x_M) = M$   
 $\exists x_m: f(x_m) = m$
  - b.  $f([a, b]) = [m, M]$

### Intermediate Value Theorem- II Version (Strong Version)

$f$  continuous on  $[a, b]$

$\Rightarrow f$  assumes all values between  $m$  and  $M$

### Fermat's Theorem

$f$  differentiable at  $x_0$

$x_0$  is an extremum point for  $f$

$$\Rightarrow f'(x_0) = 0$$

### Rolle's Theorem

$f$  continuous on  $[a, b]$

$f$  differentiable on  $(a, b)$

$$f(a) = f(b)$$

$$\Rightarrow \exists x_0 \in (a, b): f'(x_0) = 0$$

### Lagrange's Theorem

$f$  continuous on  $[a, b]$

$f$  differentiable on  $(a, b)$

$$\Rightarrow \exists x_0 \in (a, b): f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

### II Finite Increment Formula

(other way to describe Lagrange's Theorem)

$f$  differentiable on an open interval  $I$

$$x_1, x_2 \in I$$

$$x_1 < x_2$$

$$\exists t \in (x_1, x_2): f'(t) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow f(x_2) - f(x_1) = f'(t)(x_2 - x_1)$$

$$\Delta f = f'(t)\Delta x$$

## Cauchy's Theorem

$f, g$  continuous on  $[a, b]$

$f, g$  differentiable on  $(a, b)$

$$g(x) \neq 0$$

$$\Rightarrow \exists x_0 \in (a, b): \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

## De L'Hôpital's Theorem

$f, g$  differentiable in  $I(\gamma) \setminus \{\gamma\}$

$$g'(x) \neq 0 \text{ in } I(\gamma) \setminus \{\gamma\}$$

$$\lim_{x \rightarrow \gamma} f(x) = \lim_{x \rightarrow \gamma} g(x) = \begin{cases} 0 \\ \infty \end{cases}$$

$$\exists \lim_{x \rightarrow \gamma} \frac{f'(x)}{g'(x)} = \lambda \in \mathbb{R}^*$$

$$\Rightarrow \lim_{x \rightarrow \gamma} \frac{f(x)}{g(x)} = \lambda$$

## Convex vs Concave

$$t(x) = f(x_0) + f'(x_0)(x - x_0)$$

$f$  differentiable at  $x_0$

$f$  convex at  $x_0$  if

$$\exists I_r(x_0) \subseteq \text{dom}f: f(x) \geq t(x) \forall x \in I_r(x_0)$$

$f$  concave at  $x_0$  if

$$\exists I_r(x_0) \subseteq \text{dom}f: f(x) \leq t(x) \forall x \in I_r(x_0)$$

$f$  strictly convex at  $x_0$  if

$$\exists I_r(x_0) \subseteq \text{dom}f: f(x) > t(x) \forall x \in I_r(x_0)$$

$f$  strictly concave at  $x_0$  if

$$\exists I_r(x_0) \subseteq \text{dom}f: f(x) < t(x) \forall x \in I_r(x_0)$$

## Rules for $o(f)$ ( $x \rightarrow 0$ )

$$o(x^n) \pm o(x^m) = o(x^{\min(n,m)})$$

$$o(\lambda f) = \lambda o(f) = o(f)$$

$$\varphi(x)o(f) = o(f), \varphi(x) \text{ bounded in } I(0)$$

$$x^m o(x^n) = o(x^{n+m})$$

$$o(x^m)o(x^n) = o(x^{n+m})$$

$$(o(x^n))^m = o(x^{nm})$$

$$(1+x)^\alpha = 1 + \alpha x + o(x)$$

## Peano's Remainder

$f$   $n$  times differentiable at  $x_0$

$\Rightarrow \exists$  one and only one polynomial  $Tf_{n,x_0}(x) : f(x) = Tf_{n,x_0} + o((x-x_0)^n)$  ( $x \rightarrow x_0$ )

$o((x-x_0)^n)$  is the remainder in Peano's form

## Conditions for integrability on $[a, b]$ (Any)

$f$  continuous on  $[a, b]$

$$f \in C^0((a, b))$$

$f$  monotone on  $[a, b]$

Set of integrable functions on  $[a, b]$ :

$$R([a, b])$$

$f$  integrable on  $[a, b]$

$\Rightarrow f$  integrable on  $[c, d] \subseteq [a, b]$

$f$  integrable on  $[a, b]$

$\Rightarrow |f|$  integrable on  $[a, b]$

$$\left| \int_{[a,b]} f \right| \leq \int_{[a,b]} |f| \quad \forall f \in R([a, b])$$

### Integral Average

$$m(f; a, b) = \frac{1}{b-a} \int_a^b f(x) dx$$

### Fundamental Theorem of Integral Calculus

*f continuous on interval I*

$$x_0 \in I$$

$$\Rightarrow F_{x_0}(x) = \int_{x_0}^x f(t) dt \text{ is differentiable on } I$$

$$\frac{d}{dx}(F_{x_0}(x)) = f(x)$$

### Torricelli-Barrow Theorem

*f continuous in I*

*F(x) is any primitive of f on I*

$$a, b \in I$$

$$\Rightarrow \int_a^b f(x) dx = F(b) - F(a) = [F(x)]_a^b = F(x)|_a^b$$

### Integration By Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

## Asymptotic Comparison Theorem

$$f \in R_{loc}([a, +\infty])$$

$$f(x) \geq 0 \forall x \in [a, +\infty)$$

1.

$$\exists \alpha > 1: f(x) = O\left(\frac{1}{x^\alpha}\right) (x \rightarrow +\infty)$$

$$\Rightarrow \int_a^{+\infty} f(x) dx = l \in \mathbb{R}$$

(convergent)

2.

$$\exists \beta \leq 1: f(x) = O\left(\frac{1}{x^\beta}\right)$$

$$\Rightarrow \int_a^{+\infty} f(x) dx = \pm\infty$$

(divergent)

## Absolute Convergence Theorem

$$f \in R_{loc}([a, +\infty))$$

$$\int_a^{+\infty} |f(x)| dx = l \in \mathbb{R}$$

$$\Rightarrow \int_a^{+\infty} f(x) dx = k \in \mathbb{R}$$

$$\int_a^{+\infty} |f(x)| dx \geq \left| \int_a^{+\infty} f(x) dx \right|$$

## Synthetic Division of Polynomials

Use the root associated with the divisor.

$x + 3 = 0$   
 $x = -3$

$(2x^3 - 5x^2 - x + 3) \div (x + 3)$

MathBits.com

list the coefficients only

-3	2	-5	-1	3
multiply	2	-6	33	-96
	2	-11	32	-93

remainder

**Solution:**  $2x^2 - 11x + 32 + \frac{-93}{x+3}$

## Solving Separable Variable Differential Equations

$$y' = h(x)g(y)$$

$$\frac{y'}{g(y)} = h(x)$$

Integrate both sides with respect to  $x$ .

## Solving Homogeneous (in $x, y$ ) Differential Equations

$$y' = f\left(\frac{y}{x}\right)$$

$$y' = f(z), z = \frac{y}{x}$$

$$\therefore y = zx$$

$$y' = z'x + z$$

$$f(z) = z'x + z$$

$$z' = \frac{f(z) - z}{x}$$

Separable variable equation

## Solving 1<sup>st</sup> Order Homogeneous ODEs

$$y' + a(x)y = 0$$

$$y' = ya(x)$$

Separable

## Solving 1<sup>st</sup> Order Non-Homogeneous/Complete ODEs

$$y' + a(x)y = b(x)$$

$$y = f(x) + g(x): f'(x) + a(x)f(x) = 0$$

### Lagrange's Variation of Parameters

$$g(x) = k(x)e^{-A(x)}: A'(x) = a(x)$$

$$g'(x) + a(x)g(x) = b(x)$$

$$(k(x)e^{-A(x)})' + a(x)(k(x)e^{-A(x)}) = b(x)$$

$$k'(x)e^{-A(x)} - k(x)a(x)e^{-A(x)} + a(x)k(x)e^{-A(x)} = b(x)$$

$$k'(x)e^{-A(x)} = b(x)$$

$$k(x) = \int b(x)e^{A(x)} dx$$

$$g(x) = \int b(x)e^{A(x)} dx \cdot e^{-A(x)}$$

$$\therefore y = \int b(x)e^{A(x)} dx \cdot e^{-A(x)} + f(x): f'(x) + a(x)f(x) = 0$$

## Solving Linear ODEs of II Order

$$y'' + a(x)y' + b(x)y = p(x) \quad a, b, p \in C^0(I)$$

Solution is sum of:

1. General integral of the associated homogeneous equation
2. A particular integral of the complete equation

## Solving Associated Homogeneous Equation

$$y'' + a(x)y' + b(x)y = 0$$

Solution is in the form

$$c_1\varphi_1(x) + c_2\varphi_2(x) \quad c_1, c_2 \in \mathbb{R}, \exists x \in I : \varphi_1(x) \neq k\varphi_2(x)$$

$$y'' + ay' + by = 0 \quad a, b \in \mathbb{R}$$

$$y = e^{\lambda x}$$

$$y' = \lambda e^{\lambda x}$$

$$y'' = \lambda^2 e^{\lambda x}$$

$$e^{\lambda x}(\lambda^2 + a\lambda + b) = 0$$

$$\lambda^2 + a\lambda + b = 0$$

Solve quadratic (characteristic equation)

$$a^2 - 4b = 0$$

$$\Rightarrow y = (c_1 + c_2x)e^{\lambda x}$$

## Solving Particular Solution of the Complete Equation

1. Lagrange's method
2. Similarity method

$$p(x) = p_n(x)e^{\mu x}$$

Where  $p_n(x)$  is a polynomial of degree  $n$  and  $\mu \in \mathbb{C}$ .

$$\bar{y} = q_n(x)e^{\mu x}, \mu \neq \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$$\bar{y} = xq_n(x)e^{\mu x}, \mu = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$$\bar{y} = x^2 q_n(x) e^{\mu x}, \mu = \frac{-a + \sqrt{a^2 - 4b}}{2} = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

Case  $p(x)$  is a Trigonometric Function

e.g.

$$y'' + y = t \cos t$$

$$y'' + y = \frac{t}{2} e^{it} + \frac{t}{2} e^{-it}$$

Solution of equation is:

$$y = y_1 + y_2$$

Where:

$$y_1'' + y_1 = \frac{t}{2} e^{it}$$

$$y_2'' + y_2 = \frac{t}{2} e^{-it}$$

Cauchy Problem Details

$$y' = f(y, x)$$

$$y(x_0) = y_0, x \in I$$

$f$  continuous

$\Rightarrow y$  admits at least one local (to  $I$ ) solution

In addition, to verify if the solution is unique, the following condition (Lipschitz condition) must be met:

$$|f(x, y) - f(x, \tilde{y})| \leq l |y - \tilde{y}| \quad \forall x \in I$$