Linear Algebra and Geometry

Table of Contents

1: Geometric vectors in the plane and in space

- **Applied vectors** are applied onto a point.
  - Application point + free vector
- **Free vectors** are defined by:
  - Direction
  - Magnitude
  - Verse
- Zero vector ($\vec{0}$) does not have direction or verse
- Vector of magnitude 1 is called a versor or unit vector

2: More operations with vectors: scalar product, vector product and mixed product

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \theta$$

- Let $\vec{v}, \vec{w} \neq \vec{0} \in \mathbb{R}^n$
  $\vec{z}$, the projection of $\vec{u}$ along the axis of $\vec{w}$ is defined by:
  $$\vec{z} = \frac{(\vec{u} \cdot \vec{w})\vec{w}}{|\vec{w}|^2}$$

$$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin \theta$$

- Right hand rule:
  - $\vec{v}$: index finger
  - $\vec{w}$: middle finger
  - $\vec{v} \times \vec{w}$: thumb
- Properties where $\vec{u}, \vec{v}, \vec{z} \in \mathbb{R}^3, a \in R$:
  - $(a\vec{u}) \times \vec{v} = a(\vec{u} \times \vec{v})$ (Linearity 1)
  - $(\vec{u} + \vec{v}) \times \vec{z} = \vec{u} \times \vec{z} + \vec{v} \times \vec{z}$ (Linearity 2)
  - $(\vec{u} \times \vec{v}) = -(\vec{v} \times \vec{u})$ (Skew symmetry)
  - $\vec{u} \times \vec{u} = \vec{0}$
- Mixed product $= |\vec{u} \times \vec{v} \cdot \vec{z}|$
  - Volume of box defined by the 3 vectors

3: Planes and Lines

- Parametric form of an equation of a plane:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{p} + \lambda\vec{v_1} + \mu\vec{v_2}$$

Normal ($\vec{n}$) and point ($\vec{v_0}$) form:

$$(\vec{v} - \vec{v_0}) \cdot \vec{n} = 0$$

$$\vec{v} \cdot \vec{n} = \overrightarrow{v_0} \cdot \vec{n}$$

4: More on Planes and Lines

- Given a plane π, defined by $\vec{v} \cdot \overrightarrow{n_\pi} = \vec{O} \cdot \overrightarrow{n_\pi}$, the orthogonal projection of a point P on π is the point H such that

$$\overrightarrow{HP} = \overrightarrow{OP} \cdot \overrightarrow{n_\pi} \frac{\overrightarrow{n_\pi}}{|\overrightarrow{n_\pi}|^2}$$

or

$$\overrightarrow{HP} = (\overrightarrow{OP} \cdot \overrightarrow{n_\pi})\overrightarrow{n_\pi}$$

If $|\overrightarrow{n_\pi}| = 1$

5: Matrix Addition and Multiplication

- Matrix is rectangular array of numbers
  - Each Number is called an entry, element or component of the matrix
- Dimensions of matrix are given by $height \times width$
  - $A \in \mathbb{R}^{height,width}$
- Transposition of a matrix is inverting rows and columns, so rows become columns and vice-versa
  - A^T or ${}^tA$
  - Transposition is self-inverse
  - $A^T: \mathbb{R}^{m,n} \to \mathbb{R}^{n,m}$
- Adding 2 matrices means adding 2 matrices of the same dimensions component-wise.
- Multiplying a matrix by a scalar c means multiplying all entries by c
- A null matrix is denoted by **0** or 0
- $$AB = \begin{pmatrix} \leftarrow \mathbf{r}_1 \rightarrow \\ \cdots \\ \leftarrow \mathbf{r}_m \rightarrow \end{pmatrix} \begin{pmatrix} \uparrow & \cdot & \cdot & \uparrow \\ \mathbf{c}_1 & \cdot & \cdot & \mathbf{c}_p \\ \downarrow & \cdot & \cdot & \downarrow \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 \cdot \mathbf{c}_1 & \cdot & \cdot & \mathbf{r}_1 \cdot \mathbf{c}_p \\ \cdots & & & \cdots \\ \mathbf{r}_m \cdot \mathbf{c}_1 & \cdot & \cdot & \mathbf{r}_m \cdot \mathbf{c}_p \end{pmatrix}.$$
  - Rule for dimensions of multiplied matrices
    - $$m \times \underbrace{n \quad n} \times p \quad \rightsquigarrow \quad m \times p,$$

6: Determinants And Inverses

- Diagonal matrices are defined by:

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} a_{x,y} = \begin{cases} k \in \mathbb{R}, x = y \\ 0, x \neq y \end{cases}$$

- Identity matrices are diagonal matrices where all non-zero terms are equal to 1
  - Identity matrix of order n (denoted by I_n) is one of square matrix of size $n \times n$
- Matrices can be raised to powers of natural numbers, which is equivalent to multiplying them by themselves
  - Powers of negative numbers are defined via the inverse

- $A \in R^{n,m}$, A^{-1} is the inverse of A

- For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$\therefore ad - bc \neq 0$$
$$ad - bc = \det(A) = |A|$$

- If $\det(A) \neq 0$, A is called invertible or non-singular
- $(A^{-1})^T = (A^T)^{-1}$
- $(AB)^{-1} = B^{-1}A^{-1}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Then one copies down the first two columns to form the extended array

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}.$$

The formula of *Sarrus* asserts that the determinant of A is the sum of the products of entries on the three downward diagonals ↘ minus those on the three upward diagonals ↗.

Equivalently,

- $$\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \quad (4)$$

- $$\vec{u} \times \vec{v} \cdot \vec{w} = \det\begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}.$$

- To find $\det(A)$, $A \in \mathbb{R}^{3,3}$:
  - Let $A_{ij} \in \mathbb{R}^{2,2}$ be the matrix obtained by removing the row i and column j from A
  - Let the components of $\tilde{A}$ be defined by:

$$a_{ij} = (-1)^{i+j}\det(A_{ij})$$

  - $A\tilde{A}^T = \det(A)\, I_n$

    If $\det(A) \neq 0$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)}\tilde{A}^T$$

- The above method can also be used for 2×2 matrices, where $\det(A) = a$, $A \in \mathbb{R}^{1,1} = (a), a \in \mathbb{R}$

7: Linear System of Equations

- A linear system can be expressed as:
  - $\begin{cases} x + 2y = 7 \\ 3x + 4y = 8 \end{cases}$
  - $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$
    - Matrix form
  - $x\begin{pmatrix} 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$
    - Column vector form

8: Row Equivalence

- Elementary row operations for matrices are:
  - Swap rows
  - Multiply a row by a non-zero constant
  - Add a multiple of a row to another

- EROs can be described as:
  - $r_i \leftrightarrow r_j$
  - $r_i \longmapsto kr_i,\ k \neq 0$
  - $r_i \longmapsto r_i + kr_j,\ i \neq j$
- In $2 \times n$ matrices, EROs can be expressed by multiplying the target matrix by an elementary matrix, defined in the following:
  - $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$
  - $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & o \\ 0 & a \end{pmatrix}$
  - $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Generally, elementary matrices are constructed by applying the target ERO to the relevant identity matrix
- An $n \times n$ matrix E is defined to be an elementary matrix if it can be constructed by applying an ERO to I_n
- Elementary matrices are invertible since EROs are invertible
- Given an ERO, and the elementary matrix obtained by such ERO E, EA is the result of applying the same ERO to A
- $(AB)^{-1} = B^{-1}A^{-1}$
  - Notice the order is inverted
- EROs can be used to simplify systems of equations to them solve them
  - They should first be rearranged to resemble a triangular matrix as much as possible, the solution should then be clear
- Assume the solution to a system of equations is:
  - $(x_1, x_2, x_3, x_4) = (-s - t, -s - 2t, s, t)$
  - Or $\vec{x} = \begin{pmatrix} -s - t \\ -s - 2t \\ s \\ t \end{pmatrix}$
  - The set of solutions is therefore the subspace:
    - $\mathcal{L}\{u, v\}\ u = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, v = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$
- $A \sim B$ means that B is achieved by applying one or more EROs to A
  - A and B are row equivalent
- In summary:
  - If $A \sim B$, then $\vec{x}$ is a solution of $A\vec{x} = \underline{0}$ iff $B\vec{x} = \underline{0}$

9: Reduced Matrices and Their Rank

- The previously discussed method of simplifying systems of equations using EROs is called Gaussian Elimination
- An entry of a matrix is called a marker if it is the first non-zero term of a row, starting from the left.
- A step-reduced matrix is one that follows the following:

    - Every column has at most 1 maker
    - Moving down rows, the markers will move right
    - The bottom rows are the null rows
- A super-reduced matrix also sustains the following:
    - Every marker equals 1
    - The marker is the only non-zero term in its column
- When a matrix has been reduced, it is in echelon form
    - Every row is linearly independent from the others
    - Definition of linear independence:
        - Given a set $A \subset \mathbb{R}^n = \{\overrightarrow{u_1}, \overrightarrow{u_2} \dots \overrightarrow{u_n}\}$
        - $x_1\overrightarrow{u_1} + x_2\overrightarrow{u_2} + \cdots + x_n\overrightarrow{u_n} = 0$ only has solution $x_1 = x_2 = \cdots = x_n = 0$
- A set $\{\overrightarrow{u_1}, \overrightarrow{u_2} \dots \overrightarrow{u_n}\}$ is LI iff no element can be expressed as the linear combination of others
    - A singleton set $\{\vec{v}\}$ is LI iff $\vec{v} \neq \vec{0}$
    - If $\vec{0} \in A$, then the set is not LI
- Let B be a step-reduced matrix
    - ⇒It's non-zero rows are LI
- Let B be a step-reduced matrix
    - A column c_j is unmarked iff it is a LC of the previous columns
    - Or null if $j = 1$
- If B is step reduced, then its marked columns are LI
- The rank of a matrix is the number of linearly independent rows or columns in any matrix
    - Let B be step-reduced.
        - $\Rightarrow rank\ B = rk\ B = r(B)$ is equal to the number of markers
    - If $r(B) = \min(n, m) \Rightarrow B$ has full rank, or maximal rank
- $B \sim C$
    - $\Rightarrow r(B) = r(C)$
- Any matrix is row equivalent to a super-reduced one

10: Solving a General System

$$A\vec{x} = \vec{b}, A \in \mathbb{R}^{n,m}, \vec{x} \in \mathbb{R}^{1,n}, \vec{b} \in \mathbb{R}^{1,m}$$

$$(A|\vec{b}) = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

- The bar dividing A and $\vec{b}$ is purely for clarity, indicating that the final column is significant, however, it should not be considered in calculations.
- The m rows of $A \in \mathbb{R}^{n,m}$ are LI iff $r(A) = m$
- $r(A) < r(A|b)$
  ⇒The system has no solution

- $r(A) = r(A|b)$
  ⇒The system has solutions with $n - r$ parameters.
  If $n = r$, there is a unique solution (zero parameters)
  - In the case of parameterized solutions, we say that the system has ∞^{n-r} solutions
- If $A \in \mathbb{R}^{n,n}$ is invertible, then the unique super-reduced form of $(A|I_n)$ is $(I_n|A^{-1})$

11: Rank, inverse, and determinant (part II)

- $A \in \mathbb{R}^{n,n}$, A is invertible iff $r(A) = n$
- Upper triangular matrices are ones with non-zero entries above, while lower triangular matrices have non-zero entries below
  - Diagonal matrices are both
- Calculating determinant of any square matrix:

$$A \in \mathbb{R}^{n,n}$$
$$A_{ij} \in \mathbb{R}^{n-1,n-1}$$

  Is the submatrix obtained by removing the i-th row and the j-th column from A.

$$\det(A) = \sum_{i=1}^{n} (-1)^{1+i} a_{1i} \det(A_{1i})$$
$$n = 1 \Rightarrow \det(A) = a_{11}$$

  - $(\mathrm{n} = 2 \Rightarrow \det(A) = a_{11}a_{22} - a_{12}a_{21})$
- If $A \in \mathbb{R}^{n,n}$ has a zero row, $\Rightarrow \det(A) = 0$
- $A, B \in \mathbb{R}^{n,n}, \det(AB) = \det(A)\det(B)$
  - Let E_i be the elementary matrix of EROs:
    1. $r_i \mapsto r_i + ar_j, i \neq j$
    2. $r_i \mapsto cr_i, c \neq 0$
    3. $r_i \leftrightarrow r_j, i \neq j$
  - $\det(E_1) = 1$
  - $\det(E_2) = c$
  - $\det(E_3) = -1$
- If T is an upper diagonal matrix, the determinant is the product of the diagonal elements
- Laplace's Theorem:
  - $A \in \mathbb{R}^{n,n}$
    1. $\det(A) = \sum_{i=0}^{n} (-1)^{i+i} a_{ji} \det(A_{ji})\ \forall j$
    2. $\sum_{i=0}^{n} (-1)^{i+i} a_{ji} \det(A_{li}) = 0\ \forall j, l : j \neq l$

12: Subspaces of $\mathbb{R}^n$

- V is a subspace iff:
  - (S1) $\vec{u}, \vec{v} \in V \Rightarrow \vec{u} + \vec{v} \in V$,
  - (S2) $\vec{u} \in V, a \in \mathbb{R} \Rightarrow a\vec{u} \in V$
- $\{\vec{0}\}$ (aka null subspace) is the only vector subspace with 1 element, all others have infinite elements
  - All subspaces must contain zero-vector
- $\mathbb{R}^n$ is also a subspace

- Any subset $\mathcal{L}\{u_1, u_2 \dots u_n\}, u_i \in \mathbb{R}^n$ is a subspace
- Solutions to a linear homogeneous system is always a subspace
- $Ker(A) = \{\vec{x} \in \mathbb{R}^{n,1} : A\vec{x} = \mathbf{0}\}$
    - Aka kernel or null space of matrix A
- Row space of A: $Row(A) = \mathcal{L}\{\boldsymbol{r_1}, \boldsymbol{r_2} \dots \boldsymbol{r_m}\} \subset \mathbb{R}^{1,n}$
- Column space of A: $Col(A) = \mathcal{L}\{c_1, c_2 \dots c_n\} \subset \mathbb{R}^{m,1}$
- $Ker(A) = \{\boldsymbol{x} \in \mathbb{R}^{n,1} : \boldsymbol{rx} = 0\ \forall \boldsymbol{r} \in Row(A)\}$
- $A \sim B$
    - $\Leftrightarrow Row(A) = Row(B)$
    - $\Leftrightarrow Ker(A) = Ker(B)$

13: Bases and Dimensions

- Let L be a subspace of $\mathbb{R}^n$. A basis of V is a LI set $\mathcal{B} = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3} \dots \boldsymbol{v_k}\}$: $\boldsymbol{V} = \boldsymbol{\mathcal{L}}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3} \dots \boldsymbol{v_k}\}$
- The non-zero rows of a step-reduced matrix A form a basis of $Row(B)$
- Let V be a subspace of $\mathbb{R}^n$ that is not null
    - $\Rightarrow$ The basis of V has at most n elements
    - $\Rightarrow$ Any 2 bases of V have the same number of elements
- The number of elements in a basis of V is called the dimension of V
    - $= \dim V$
- $\forall A \in \mathbb{R}^{n,m}$
    - $r(A) = \dim(Row(A))$
    - $\dim Row(A) = \dim Ker(A) = n - r(A)$
    - $\dim(Col(A)) = r(A)$
    - $r(A) = r(A^T)$

14: Vector Spaces

- $\mathbb{F}$ is a field of scalars
    - Examples are $\mathbb{R}, \mathbb{C}, \mathbb{Q}, B = \{0,1\}$
    - Multiplication between elements of fields must be commutative: $ab = ba$
    - There must be a multiplicative identity or unit (1), and a multiplicative inverse such that $aa^{-1} = 1, a \neq \mathbf{0}$
    - There must be an additive identity (0) such that $a + 0 = a$
    - Every field must contain at least an additive identity and a multiplicative identity
- V is a vector space on $\mathbb{F}$ if:
    - $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$
    - $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
        - $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$
    - $\mathbf{0} + \boldsymbol{v} = \boldsymbol{v}\ \forall \boldsymbol{v} \in V$
    - $\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}$
    - $(a + b)\boldsymbol{v} = a\boldsymbol{v} + b\boldsymbol{v}$
    - $(ab)\boldsymbol{v} = a(b\boldsymbol{v})$

- $a(\boldsymbol{u}+\boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}$
- $1\boldsymbol{v} = \boldsymbol{v}$
  - $\forall a, b \in \mathbb{F}$
- When $\mathbb{F} = \mathbb{R}$, V is called the real vector space
- The set $\mathbb{F}_n[x]$ of polynomials in variable x with no more than n coefficients in a field $\mathbb{F}$ is a vector space over $\mathbb{F}$
- Let V be a vector space and $\mathcal{A}$ a non-empty set. The set of all mappings $f: \mathcal{A} \to V$ is a vector space
  - This is because the functions can be added and multiplied by scalars

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15: Bases and Linear Mappings

- Let $u_1, u_2 \dots u_n \in V$ where V is a vector space on F,
  - $\mathcal{L}(u_1, u_2 \dots u_n) = \{a_1u_1 + a_2u_2 + \dots + a_nu_n: a_k \in F\}$
  - Note that the coefficients are part of F, rather than the usual $\mathbb{R}$
- Let V be a vector space over a field F. A non-empty subset U of V is a subspace iff
  - $\boldsymbol{u}, \boldsymbol{v} \in U \Rightarrow \boldsymbol{u} + \boldsymbol{v} \in U$
  - $a \in F, \boldsymbol{u} \in U \Rightarrow a\boldsymbol{u} \in U$
- A vector space V is finite-dimensional or finitely-generated, if there exists a finite subset $\{\boldsymbol{u_1}, \boldsymbol{u_2} \dots \boldsymbol{u_n}\}$ such that $V = \mathcal{L}(\boldsymbol{u_1}, \boldsymbol{u_2} \dots \boldsymbol{u_n})$.
- $\{\boldsymbol{v_1}, \boldsymbol{v_2} \dots \boldsymbol{v_n}\} \subset V$ is LI iff
  - $a_1\boldsymbol{v_1} + a_2\boldsymbol{v_2} + \dots + a_n\boldsymbol{v_n} = \boldsymbol{0} \Rightarrow a_1 = a_2 = \dots = a_n = 0$
- A finite set $\{\boldsymbol{v_1}, \boldsymbol{v_2} \dots \boldsymbol{v_n}\} \subset V$ is a basis of V if
  - $V = \mathcal{L}\{\boldsymbol{v_1}, \boldsymbol{v_2} \dots \boldsymbol{v_n}\}$
  - The set is LI
- Let V, W be two vector spaces over F
  - A mapping $f: V \to W$ is called linear if $\forall a \in F$ and $\boldsymbol{u}, \boldsymbol{v} \in V$:
    - $f(\boldsymbol{u}+\boldsymbol{v}) + f(\boldsymbol{u}) + f(\boldsymbol{v})$
    - $f(a\boldsymbol{u}+v) = af(\boldsymbol{u}) + f(\boldsymbol{v})$
  - Consequence: $f(\boldsymbol{0}) = \boldsymbol{0}$
- A bijective linear mapping is called an isomorphism
- Let V be a vector space with a basis of size n (V has dimension n)
  - If m vectors $\boldsymbol{v_1}, \boldsymbol{v_2} \dots \boldsymbol{v_m}$ of V are LI, then $m \leq n$
  - $V = \mathcal{L}\{\boldsymbol{v_1}, \boldsymbol{v_2} \dots \boldsymbol{v_p}\} \Rightarrow n \leq p$

16: Linear Mappings and Matrices

- A matrix $A \in \mathbb{R}^{m,n}$ defines a linear mapping $f: \mathbb{R}^n \to \mathbb{R}^m$
  - $f(\boldsymbol{v}) = A\boldsymbol{v}, v \in \mathbb{R}^{n,1}$
- Let V be a vector space with basis $\{v_1, v_2 \dots, v_n\}$. A linear mapping $f: V \to W$ is completely determined by vectors $f(v_1), f(v_2) \dots f(v_n)$, which can be assigned arbitrarily.
- Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. The associated matrix is $M_f \therefore f(\boldsymbol{v}) = M_f\boldsymbol{v}$
- Let $h = f \circ g$, where both f, g are linear mappings.
  - $M_h = M_fM_g$

- Let $g: V \rightarrow W$ be a linear mapping
  - $g^{-1}(\mathbf{0}) = \ker(g)$ is a subspace of V. Its dimension is called the nullity of g
  - $Im(g)$ is a subspace of W. Its dimension is called the rank of g
- Rank-Nullity Theorem
  - Given a linear mapping $g: V \rightarrow W$
    - $\dim V = \dim\big(Ker(g)\big) + \dim\big(Im(g)\big)$
    - $\dim V = nullity(g) + rank(g)$
- Given a linear mapping $g: V \rightarrow W$ with $\dim V = n, \dim W = m$
  - g injective $\Leftrightarrow nullity(g) = 0 \Leftrightarrow rank(g) = n$
  - g surjective $\Leftrightarrow rank(g) = m$
  - g bijective $\Leftrightarrow nullity(g) = 0, rank(g) = m \Leftrightarrow rank(g) = m = n$

17: Operations and Subspaces

- Let U, V be subspaces of vector space W
  - $U \cap V$ is a subspace of W
  - $U \cup V$ is a subspace of W iff $U \subseteq V$ or $V \subseteq U$
- $U + V = \{\boldsymbol{u} + \boldsymbol{v} : \boldsymbol{u} \in U, \boldsymbol{v} \in V\}$
  - It is a subspace and is contained in any subspace containing $U \cup V$
- $\dim U + \dim V = \dim(U \cap V) + \dim(U + V)$

18: Eigenvectors and Eigenvalues

- Consider $f: V \rightarrow V$ where V is a vector space with field of scalars F
- $v \neq \mathbf{0} \in V$ is called an eigenvector of f if $\exists \lambda \in \mathrm{F}: f(v) = \lambda v$
  - λ is called the eigenvalue associated to v
- $A \in \mathbb{R}^{n,n}$ has at most n LI eigenvectors
- If v is an eigenvector of A with eigenvalue λ, then it is an eigenvector of $A + aI_n$ with eigenvalue $\lambda + a$
- If v is an eigenvector of A with eigenvalue λ, then $\det(A - \lambda I) = 0$. The opposite holds.
- $p(x) = \det(A - xI)$ is the characteristic polynomial of the square matrix A of degree n in the variable x
- A is invertible iff 0 is not an eigenvalue of A

19: Eigenspaces and Multiplicities

- If λ is an eigenvalue, the subspace $E_\lambda = \ker(A - \lambda I)$ is called the eigenspace associated to λ
- Suppose $\{v_1, v_2 \dots v_k\}$ are eigenvectors with distinct eigenvalues $\{\lambda_1, \lambda_2 \dots \lambda_k\}$.
  - $\Rightarrow \{v_1, v_2 \dots v_k\}$ is LI
- If a matrix has n distinct eigenvalues, then $\mathbb{R}^n$ has a basis consisting of eigenvectors of A
- $\mathrm{mult}\,\lambda$ (the multiplicity of λ) of characteristic polynomial $p(x)$, is the highest power of the factor $x - \lambda$ that divides $p(x)$
  - $1 \leq \dim E_\lambda \leq mult(\lambda)$

20: Diagonalizability

- Two matrices $A, B \in \mathbb{R}^{n,m}$ are said to be similar if $\exists P: A = PBP^{-1}$
- A matrix $A \in \mathbb{R}^{n,n}$ is said to be diagonalizable if it is similar to a diagonal matrix
- A matrix $A \in \mathbb{R}^{n,n}$ is diagonalizable over $\mathbb{R}$ iff there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.
- A matrix $A \in \mathbb{R}^{n,n}$ is diagonalizable over $\mathbb{R}$ iff all the roots of $p(x) \in \mathbb{R}$, and for each repeated root λ we have $mult(\lambda) = \dim E_\lambda$
- A linear mapping $f: V \to V$ is called an endomorphism
- We say that an endomorphism $f: V \to V$ is simple if there exists a basis of V made by eigenvectors of f
- Let $f: V \to V$ be an endomorphism, then the following are equivalent facts:
  - f is simple
  - $\exists$ a basis of V made of eigenvectors of f
  - $\exists$ a basis of V such that the associated matrix to f is diagonal
  - $\exists$ a basis of V such that the associated matrix to f is diagonalizable
- If A, B are associated matrices to the same endomorphism f, then A, B are similar

21: Symmetric and Orthogonal Matrices

- A matrix A is symmetrical if $A = A^T$
- Let v_1, v_2 be eigenvectors of a matrix A with distinct eigenvalue λ_1, λ_2. $\Rightarrow v_1 \cdot v_2 = 0$
- Let $v_1, \ldots, v_m$ be a basis of the subspace $V \subseteq \mathbb{R}^n$. The basis is called an orthonormal basis (ON basis for short) if:
  - $v_i \cdot v_i = 1 \; \forall i$ (The length of the vectors is always 1)
  - $v_i \cdot v_j = 0 \; \forall i \neq j$ (All vectors are perpendicular to each other)
- A matrix P is called orthogonal if it satisfies one of:
  - $P^T P = I_n$
  - $PP^T = I_n$
  - $P^{-1} = P^T$
  - What the above means is that all columns are orthogonal vectors of length 1. Same applies to rows
- Let $S \in \mathbb{R}^{n,n}$ be a symmetrical matrix.
  - The eigenvalues of S are all real
  - $\exists$ an orthogonal matrix $P: P^{-1}SP = D$
    - Where P is a matrix constructed by the normalized eigenvectors of S
    - D is a diagonal matrix constructed of the eigenvalues of S
- If $S \in \mathbb{R}^{n,n}$ is symmetric, then it is also diagonalizable
- Any 2×2 orthogonal matrix with determinant 1 has the form $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and represents an anticlockwise rotation in $\mathbb{R}^2$ by an angle θ with center the origin.
- If $A, B \in \mathbb{R}^{n,n}$ are orthogonal, then A^{-1}, AB are also orthogonal
- Any orthogonal matrix has $\det A = \pm 1$

22: The Graham-Schmidt Algorithm and the Cayley-Hamilton Theorem

- Let $\{v_1, v_2 \dots v_n\}$ be an orthonormal basis of $\mathbb{R}^n$ and let $u \in \mathbb{R}^n$
    - $\Rightarrow u = u_1 v_1 + \cdots + u_n v_n$
    - $u_i = u \cdot v_i$ for $i = 1,2 \dots n$
- Let $\{v_1, \dots v_n, u\}$ be a basis of V such that $v_i \cdot v_i = 1\ \forall i, v_i \cdot v_j = 0\ \forall i \neq j$
    - $\Rightarrow \{v_1, v_2 \dots v_n, v\}$ is an orthonormal basis of V where

$$v = \frac{u'}{||u'||}$$
$$u' = u - pr_{v_1}(u) - \cdots - pr_{v_n}(u)$$
$$pr_v(u) = u \cdot \frac{v}{||v||^2} v$$

    - The Graham-Schmidt algorithm uses this method over and over for every element of the basis, starting from the second all the way to the last. (The first one is just normalized)
    - So, each vector uses this algorithm with all previous vectors being used to subtract the projections from u
- If $A = PDP^{-1}$ where D is diagnonal, then $A^n = PD^nP^{-1}$
- Cayley-Hamilton Theorem:
    - Let $A \in \mathbb{R}^{n,n}$ with characteristic polynomial $p(x) = a_0 + a_1 x + \cdots + (-1)^n x^n$
        - $\Rightarrow p(A) = a_0 I_n + a_1 A + \cdots + (-1)^n A^n = 0$

23: Quadratic Forms and Conics

- A homogeneous polynomial of degree n in $x_1, \dots x_k$ is one such that every term has degree n.
    - $n = 1$: Linear form
    - $n = 2$: Quadratic form

$$q(\vec{x}) = \vec{x}^T A \vec{x} = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

- Quadratic forms represent ellipses, hyperbolae or quadratics when having the equation $q(\vec{x}) = k, \vec{x} \in \mathbb{R}^2$
- A quadratic form $q(\vec{v}) = \vec{v}^T A \vec{v}$ is (character of definition):
    - Positive definite: $q(\vec{v}) > 0\ \forall \vec{v} \neq \vec{0} \in \mathbb{R}^n$
    - Negative definite: $q(\vec{v}) < 0\ \forall \vec{v} \neq \vec{0} \in \mathbb{R}^n$
    - Indefinite: Neither, but $q(\vec{v}) \neq 0\ \forall \vec{v} \neq \vec{0}$
    - Semi-positive/negative have $\geq, \leq$ instead
    - Signs of eigenvalues determine shape and character of definition, or by the value of

$$\Delta = B^2 - 4AC, \qquad Ax^2 + Bxy + Cy^2$$

        - ++/--: Ellipsis, +ve definite/-ve definite, $\Delta < 0$
        - +-: Hyperbola, Indefinite, $\Delta > 0$
        - +0/-0: Quadratic, $\Delta = 0$
            - $\det S = 0$

24: More on Conics

- If $S = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ represents a quadratic form
$$q(x,y) = Ax^2 + 2Bxy + Cy^2, \qquad (1)$$
then it can be diagonalized to form
$$q(x,y) = \lambda_1 X^2 + \lambda_2 Y^2, \qquad \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} (\overrightarrow{v_1} | \overrightarrow{v_2})^T$$
- The conic
$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F, \qquad (2)$$
can be achieved with the matrices
$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}, \qquad \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix}$$
  - The conic is central if there is a **translation** that can turn it into form in Equation 1
    - The origin of the original is at
$$(u,v): \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -D \\ -E \end{pmatrix}$$
    - This removes the linear terms
    - If this is possible, then the conic is called central
  - If the determinant of the 3x3 matrix is 0, then the conic is degenerate
- Quadric surface:
$$q(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
  - Can be diagonalized into
$$X^2 + Y^2 + Z^2 = -J$$
- Paraboloids:
$$Z = aX^2 + bY^2$$
  - They do not have a center