Linear Algebra Key Points

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Determinant and Inverse

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \qquad \det A = |A| = ad - bc$$

$$(A^{-1})^T = (A^T)^{-1}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$\det\begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} = \vec{u} \times \vec{v} \cdot \vec{w}$$

$$A \in \mathbb{R}^{n,n} \text{ has a zero row} \Rightarrow \det A = 0$$

$$\det(AB) = \det A \det B\,, \qquad A, B \in \mathbb{R}^{n,n}$$

Finding $\det A : A \in \mathbb{R}^{n,n}$

- Let $A_{ij} \in \mathbb{R}^{n-1,n-1}$ be the matrix obtained by removing the row i and column j from A
- Let the components of $\tilde{A}$ be defined by:

$$a_{ij} = (-1)^{i+j} \det(A_{ij})$$

- $A\tilde{A}^T = \det(A)\, I_n$

  If $\det(A) \neq 0$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)}\tilde{A}^T$$

Or

$$\det A = \sum_{i=1}^{n} (-1)^{1+i} a_{1i} \det A_{1i}$$

$$n = 1 \Rightarrow \det A = a_{11}$$

Laplace's Theorem

$$A \in \mathbb{R}^{n,n}$$

$$\det(A) = \sum_{i=0}^{n} (-1)^{1+i} a_{ji} \det(A_{ji}) \ \forall j$$

$$\sum_{i=0}^{n} (-1)^{1+i} a_{ji} \det(A_{li}) = 0 \ \forall j\,, \qquad l : j \neq l$$

What Laplace's theorem means is that $A\tilde{A}^T = \det(A)\, I_n$

Solutions of Linear Systems

Consider $A\vec{x} = \vec{b}, A \in \mathbb{R}^{n,m}, \vec{b}, \vec{x} \in \mathbb{R}^m$

$$r(A) < r(A|b) \Rightarrow \text{No solution}$$
$$r(A) = r(A|b) = r \Rightarrow n - r \text{ solutions}$$

If $A \in \mathbb{R}^{n,n}$ is invertible, then the unique super-reduced form of $(A|I_n)$ is $(I_n|A^{-1})$

$$Row(A) = \mathcal{L}\{r_1, \dots, r_m\} \subset \mathbb{R}^{1,n}$$

$$Col(A) = \mathcal{L}\{c_1, \dots, c_n\} \subset \mathbb{R}^{m,1}$$

$$Ker(A) = \left\{\vec{x} \in \mathbb{R}^{n,1}: A\vec{x} = \vec{0}\right\} = \{\vec{x} \in \mathbb{R}^{n,1}: \vec{r}\vec{x} = 0\ \forall r \in Row(A)\}$$

$$\begin{aligned} A \in \mathbb{R}^{n,m} \Rightarrow\ & r(A) \\ &= \dim\big(Row(A)\big) \\ &= n - \dim\big(Ker(A)\big) \\ &= \dim\big(Col(A)\big) \\ &= r(A^T) \end{aligned}$$

Linear Mappings

$$f: \mathbb{R}^n \to \mathbb{R}^m \text{ is linear if:}$$
$$f(a\vec{u} + \vec{v}) = af(u) + f(v)$$

$$f^{-1}\big(\vec{0}\big) = \ker f$$

$$\dim \ker f = nullity(f) = \text{dimensions lost after transformations}$$

$$\dim Im(g) = r(g) = \text{dimensions of output}$$

$$\begin{aligned} &g: V \to W, \qquad \dim V = n, \qquad \dim W = m \\ &g \text{ injective} \Leftrightarrow nullity(g) = 0 \Leftrightarrow r(g) = n \\ &g \text{ surjective} \Leftrightarrow r(g) = m \\ &g \text{ bijective} \Leftrightarrow nullity(g) = 0, r(g) = m \Leftrightarrow r(g) = m = n \end{aligned}$$

Rank-Nullity Theorem

$$g: V \to W$$
$$\Rightarrow \dim V = \dim\big(Ker(g)\big) + \dim\big(Im(g)\big) = nullity(g) + r(g)$$

Translates to dim of input = dim of output + dims lost after transformation.

Eigenshit

In this section, let $\vec{v}$ be an eigenvector of $A \in \mathbb{R}^{n,n}$ with eigenvalue λ

$$A\vec{v} = \lambda\vec{v}$$

$$\Rightarrow \det(A - \lambda I) = 0$$

$$p_A(x) = \det(A - xI)$$

$p_A(x)$ is the characteristic polynomial of A of degree n

$$\Rightarrow p(A) = 0$$

$$\lambda \neq 0 \Leftrightarrow \text{ A is invertible}$$

$$E_\lambda = \mathcal{L}\{\vec{v}_1, \dots, \vec{v}_k\} = \ker(A - \lambda I)$$

$mult(\lambda)$ is the highest power of the factor $(x - \lambda)$ that divides $p_A(x)$

$$1 \leq \dim E_\lambda \leq mult(\lambda)$$

$A, B \in \mathbb{R}^{n,m}$ are similar if $\exists P: A = PBP^{-1}$

A matrix is diagonalizable if it is similar to a diagonal matrix

$$\Rightarrow A = PDP^{-1}$$
$$P = (\overrightarrow{v_1}, \dots, \overrightarrow{v_k})$$
$$D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{pmatrix}$$
$$\Rightarrow A^n = PD^nP^{-1}$$

Orthogonal Matrices

An orthonormal (ON) basis of V is

$$\{\overrightarrow{v_1}, \dots \overrightarrow{v_n}\}: v_i \cdot v_j = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

An orthogonal matrix P has rows/columns that form an orthonormal basis.

Let S be a symmetrical matrix.

$$\Rightarrow D = P^{-1}SP$$

Orthogonal matrices represent reflection ($\det P = -1$) or a rotation ($\det P = 1$)

$$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

A, B are orthogonal $\Rightarrow$ So are A^{-1}, AB