

Fundamentals of Strength of Materials

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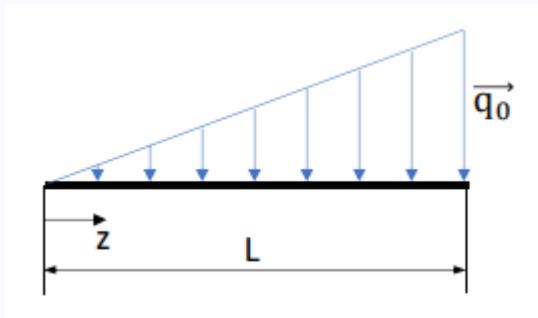
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1. Basics

1. Basics

Forces and Moments

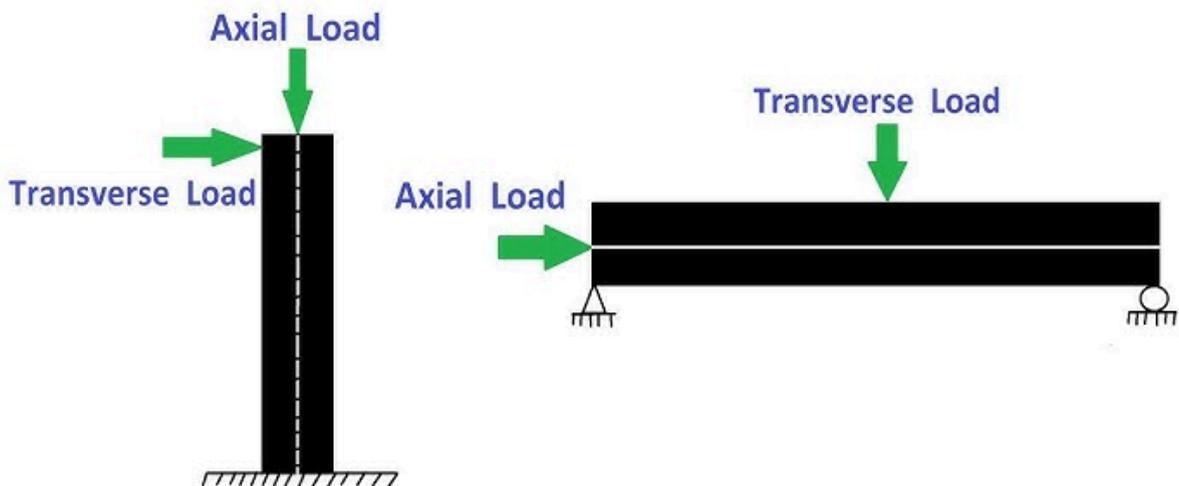
A force \vec{F} applied on a line can have a uniform or variable distribution \vec{q} . The unit for \vec{q} is Nm^{-1} .



$$\vec{q} = \vec{q}_0 \frac{z}{L}$$

$$\vec{F} = \int_0^L \vec{q}(z) dz$$

Difference between Axial Loading & Transverse Loading



Moments

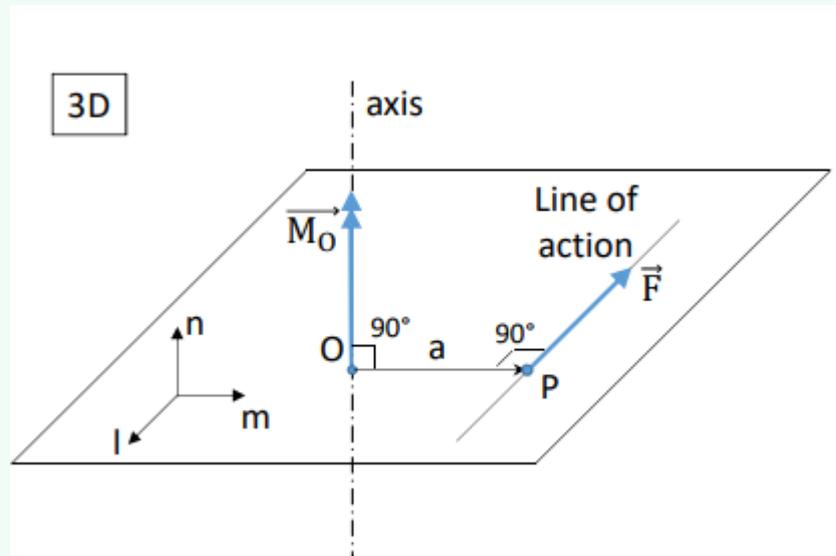
Definition

An action defined by a force that causes rotation about a given axis.

The unit is Nm

$$\vec{M} = \vec{a} \times \vec{F}$$

Where \vec{a} is called the arm — it is the position vector of the force's application point relative to the rotation's pivot.



Static equilibrium

A body is in static equilibrium when the following holds:

$$\Sigma \vec{F} = 0 \quad \Sigma \vec{M} = 0$$

Note

In 2D, that's 3 equations (2 for translation, 1 for rotation), while for 3D it's 6 equations.

Static Equivalence

Definition

Two bodies are statically equivalent if they have the same resultant force and moment.

Two forces of:

- Equal amplitude F
- Equal direction \vec{u}_F

- Opposite versus
- Different lines of actions
are statically equivalent to a moment $M = Fh$ where h is the distance between the two forces. The moment is applied in the middle between the two forces.

Any set of forces on a body are equivalent to a linearly independent set of forces and moments. This is the **equivalent force moment system**.

Determinacy of Structures

In a structure of interconnected bodies with n available degrees of freedom (DOF), and [constraints](#) that inhibit v DOFs, the degree of static determinacy is:

$$h = v - n$$

✓ What does h mean?

- $h < 0$: Not enough constraints to keep the structure static. It is therefore **statically impossible** or **kinematically indetermined**.
- $h = 0$: The structure has no free DOFs, and is therefore **statically determinated**.
- $h > 0$: There are more constraints than needed to block motion. It is therefore **statically indetermined**. However, we need h extra equations to solve the system due to forces caused by deformations. (Virtual works principle)

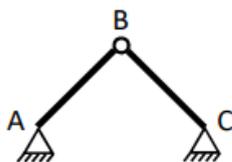
⚠ Warning

$h = 0$ does NOT guarantee that the structure is statically determinated.



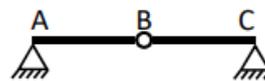
$$h = v - n = (1+1+1) - 3 = 0$$

but the horizontal translation is allowed!!!



$$h = v - n = (2+2+2) - (3+3) = 0$$

No rigid body motions.



$$h = v - n = (2+2+2) - (3+3) = 0$$

Vertical displacement of point B is allowed!!!

For the example on the bottom right, the movement might be very small, but we consider infinitesimal movement to cause kinematic indeterminacy.

Solving Structures (2D)



Definition

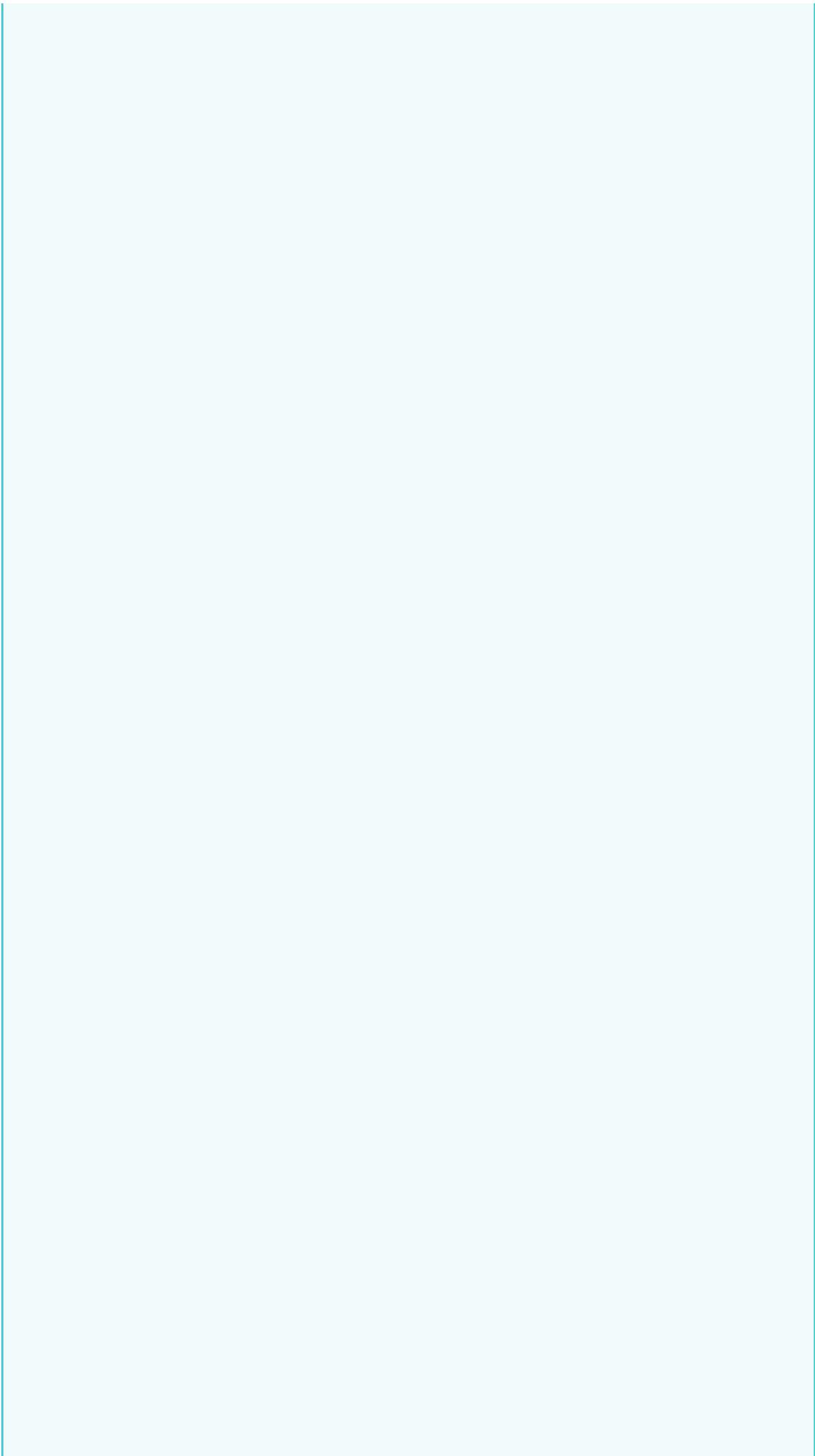


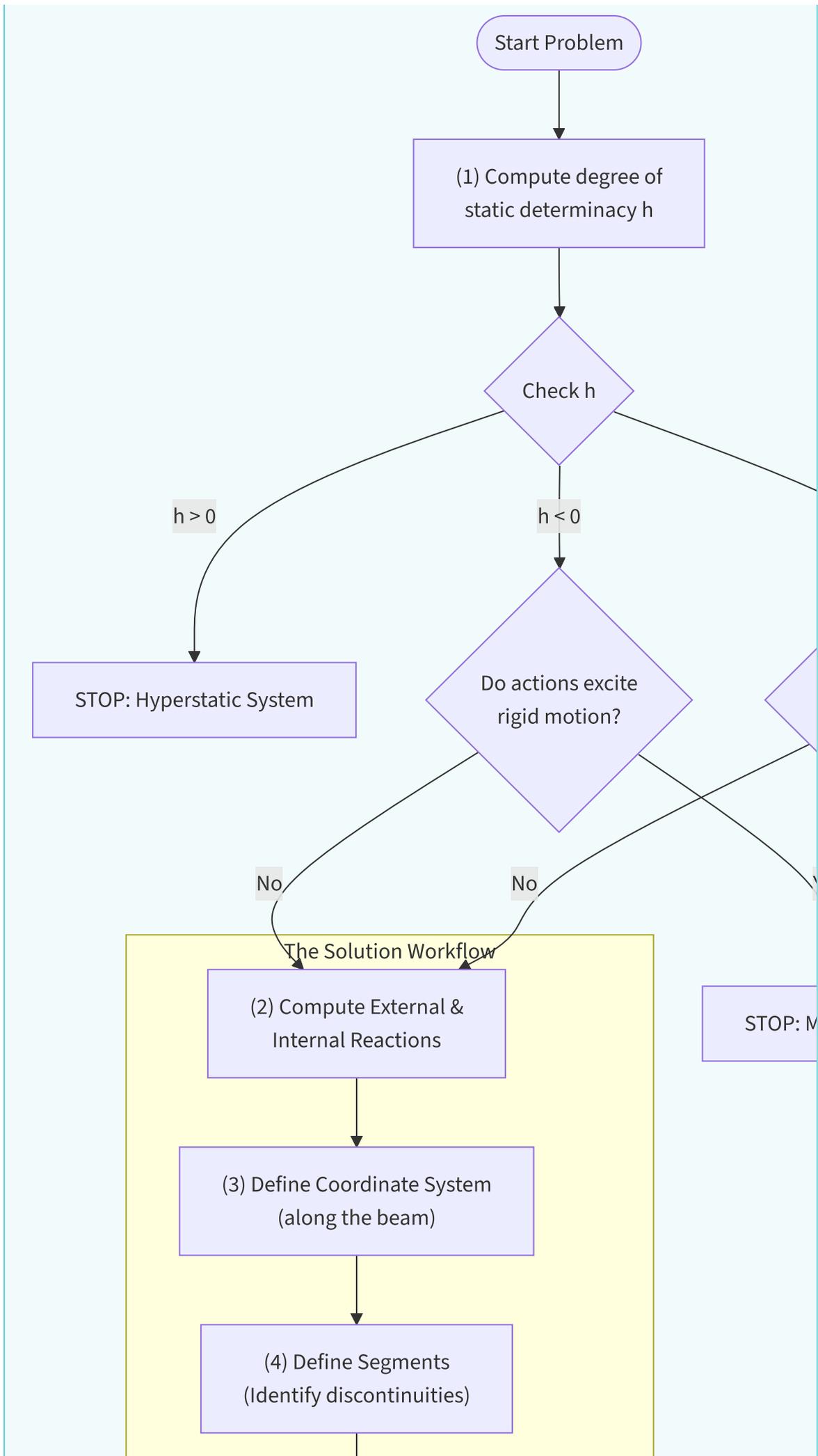
Solving a structure means calculating the unknown reaction forces (and moments) exerted by supports to maintain the body in equilibrium.

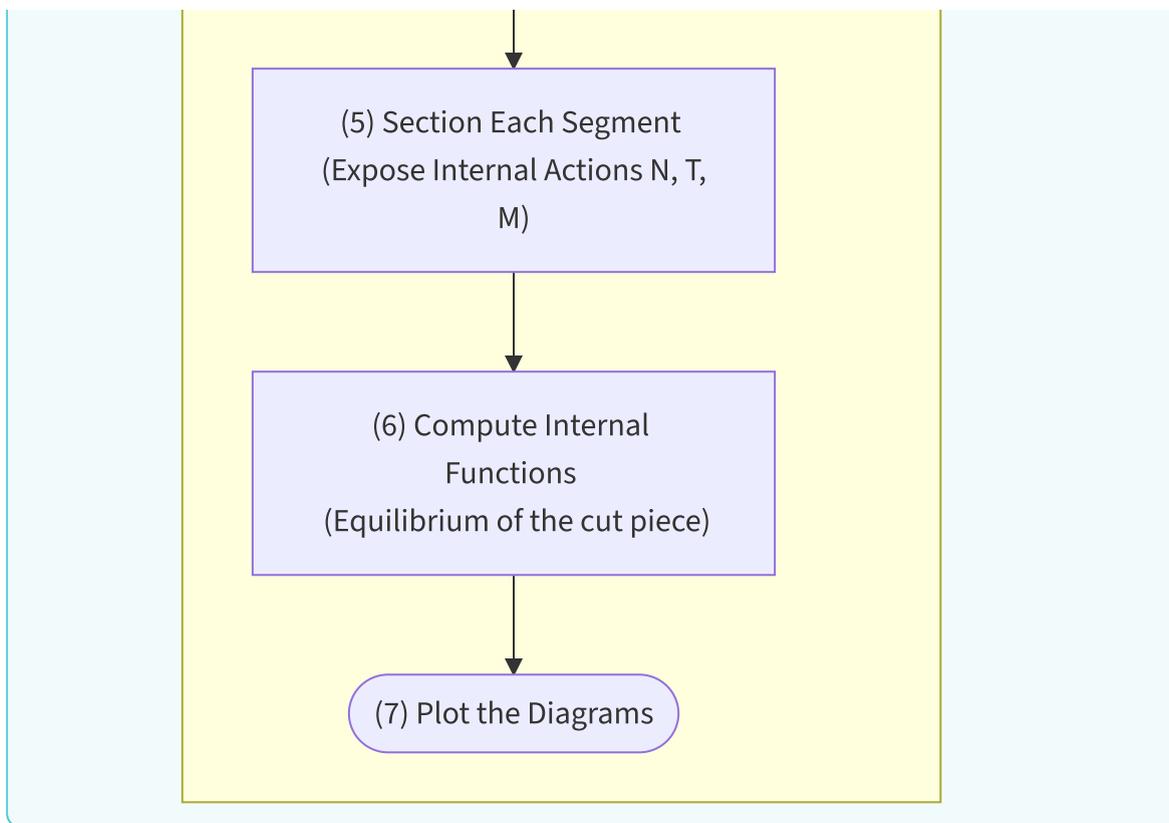


Flowgraph









1. Check Static Determinacy (h)

- $h > 0$ (Hyperstatic): STOP. Equilibrium equations are insufficient.
- $h < 0$ (Labile): STOP. Mechanism forms (unless loads avoid exciting motion).
- $h = 0$ (Isostatic): Proceed, provided no rigid motion is allowed.

2. Free Body Diagram (FBD)

- Remove supports and replace them with unknown reaction vectors (H_A, V_A, M_A).

3. Equilibrium Equations (Options)

Select one set to solve for unknowns. Tip: Aim for uncoupled equations (1 unknown per equation).

- Option 1 (Standard): 2 Forces ($\sum F_x, \sum F_y$) + 1 Moment ($\sum M_P$).
- Option 2: 2 Moments ($\sum M_P, \sum M_Q$) + 1 Force (not \perp to PQ).
- Option 3: 3 Moments ($\sum M_P, \sum M_Q, \sum M_R$) (points not aligned).

4. Assemblies (Multiple Interconnected Bodies)

When a structure consists of multiple rigid bodies connected by internal constraints (e.g., hinges), you must account for internal reactions.

A. Rule for Internal Reactions

⚡ Newton's Third Law

Internal reactions on Body #1 must be **EQUAL** and **OPPOSITE** to the internal reactions on Body #2.

- If H_B points left on Body #1, it **MUST** point right on Body #2.
- When the whole structure is re-assembled, these internal forces vanish.

B. Views for Analysis

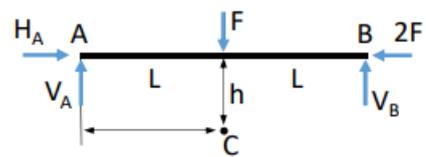
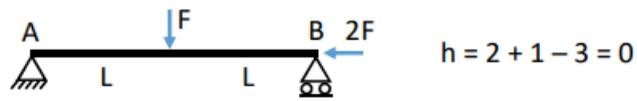
1. **Whole Structure View:** Shows only external supports. Internal hinges are hidden. Useful if total external unknowns ≤ 3 .
2. **Exploded View (Single Bodies):** Shows internal reactions at connections. Total equations available = $3 \times (\text{Number of Bodies})$.

C. Solving Strategies

You need as many equations as there are unknowns (External + Internal).

- **Option A (Hybrid):**
 - Write 3 equations for the **Whole Structure** (ignores internal forces).
 - Write 3 equations for **One Body** (reveals internal forces).
 - *Use when:* The entire structure allows you to find some external reactions immediately.
- **Option B (Exploded):**
 - Write 3 equations for **Body #1**.
 - Write 3 equations for **Body #2**.
 - (3 additional equations for every body added)
 - *Strategy:* Look for the “weak link” body (the one with only 3 unknowns) and solve that one first. Then transfer the values to the connected body.

Examples



Option #1:

Option #2:

Option #3:

Ⓐ) $V_B 2L - FL = 0$

Ⓐ) $V_B 2L - FL = 0$

Ⓐ) $V_B 2L - FL = 0$

↑ $V_A + V_B - F = 0$

Ⓑ) $-V_A 2L + FL = 0$

Ⓑ) $-V_A 2L + FL = 0$

→ $H_A - 2F = 0$

→ $H_A - 2F = 0$

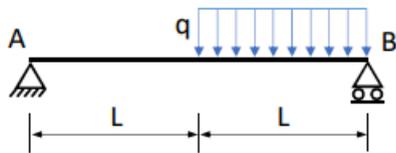
Ⓒ) $-V_A L + V_B L - H_A h + 2Fh = 0$

Note that C does not necessarily belongs to the structure

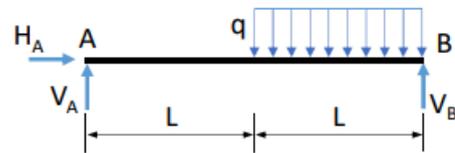
$V_B = F/2$

$V_A = F/2$

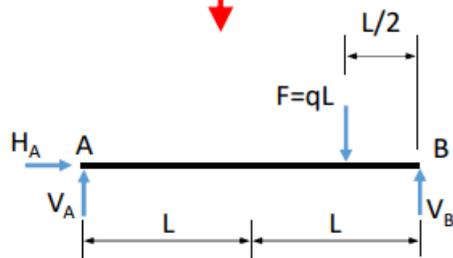
$H_A = 2F$



$h = 0$



Statically equivalent



Option #2:

Ⓐ) $V_B 2L - 3qL^2/2 = 0$

Ⓑ) $-V_A 2L + qL^2/2 = 0$

→ $H_A = 0$

$V_B = 3qL/4$

$V_A = qL/4$

$H_A = 0$

Bodies

Beams

A **Beam** is a "slender" body where the **Axial Length** is significantly larger than the **Transverse Dimensions** (cross-section).

- **Criterion:** Length \approx 10 to 30+ times the width.
- **Axis:** The line running along the axial direction.

Definition

The axis of a body with a straight axis is the locus of the **centroid of cross-sections**.

2. Constraints

2. Constraints

Symbol	Name	Constrained dofs (green) & reactions (blue)	
	(simple) support		Vertical displacement
	(simple) support		Horizontal displacement
	hinge		Vertical & horizontal displacements
	Clamp		All dofs

Internal Constraints

Constraints between dynamic bodies

Symbol	Name	Constrained dofs
	Internal hinge	Vertical & horizontal relative displacements
	Internal slide	Axial relative displacements

3. Internal Actions and Stress Resultants

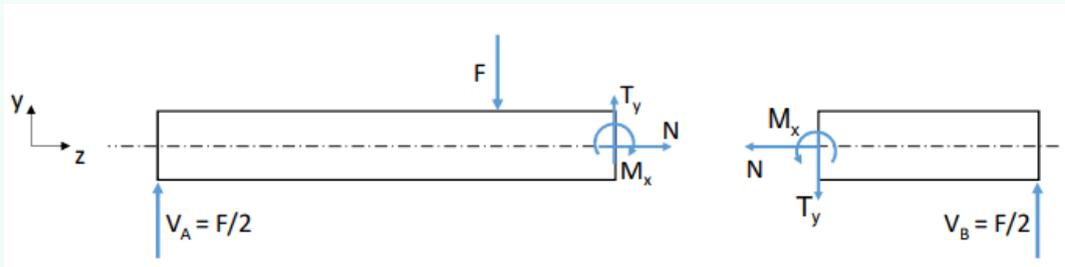
3. Internal Actions and Stress Resultants

Definition

Internal actions are unseen forces that act in the interior of a body.

They can be:

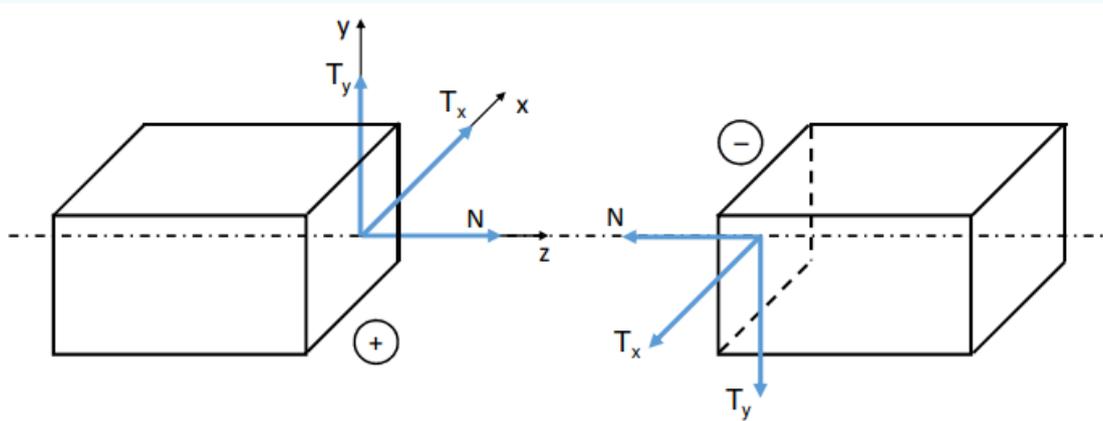
- Normal (axial) force N
- Shear (transverse) force T
- Bending moment M



To find internal actions, we use **sectioning**: cutting a body into two parts across a cross-section.

⚡ Internal actions **MUST** vanish when summed up!

Cross-section-local coordinate space conventions



A positive cross-section has beam axis z going outward the body, while a negative cross-section has the beam z -axis going inward.

The Diagrams

⚡ The Goal

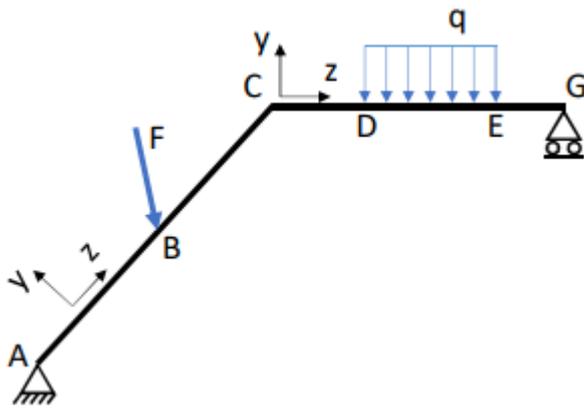
To determine the internal forces (N , T , M) at every single point along the beam's length (z). This tells us where the structure is weakest (the "Critical Points").

A. The Setup (Method of Sections)

1. **Define Coordinate System (z):** Place an origin (usually at the left end, A). The variable z travels along the beam axis.

2. **Identify Discontinuities (Segments):** You cannot write one single function for the whole beam if the loading changes. You must split the beam into segments wherever:

- A concentrated force or moment is applied (Point B).
- A constraint exists. (Points A, G)
- The geometry changes. (Point C)
- A distributed load starts or ends. (Points D, E)



B. Cutting the Beam

Step 1: Sectioning

Pick a segment (e.g., Segment AC, where $0 \leq z \leq 2a$). Imagine cutting the beam at an arbitrary distance z .

- **Rule:** Discard one half (usually the right). Keep the other (usually the left).
- **Expose Forces:** Replace the cut part with unknown internal forces: **Normal (N)**, **Shear (T)**, and **Moment (M)**.

Step 2: Equilibrium of the Cut Piece

Write equilibrium equations for *just that isolated piece* to find functions for $N(z)$, $T(z)$, $M(z)$.

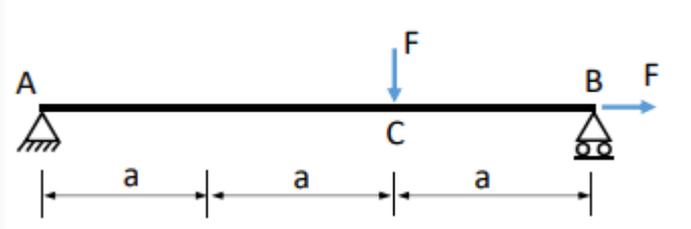
- $\sum F_z = 0 \rightarrow N$

- $\sum F_y = 0 \rightarrow T$
- $\sum M_{cut} = 0 \rightarrow M$
(Repeat for the next segment, e.g., CB)

C. The Results (The Diagrams)

 Note

The diagrams below are not related to the one above! Instead, they refer to this one:

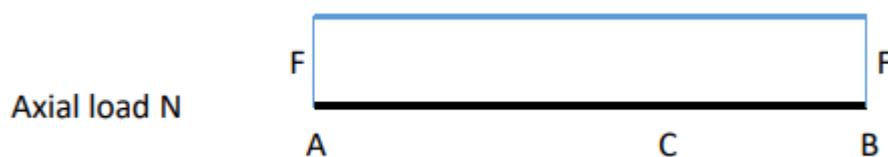


Once you have the functions, you plot them. These plots reveal the physics of the beam.

1. Axial Load Diagram (N)

Shows tension (+) or compression (-).

- In this example, it is constant ($N = F$).



2. Shear Force Diagram (T_y)

Shows the force trying to slice the beam vertically.

The Jump Rule

At Point C (where force F pushes down), the Shear Diagram **JUMPS** exactly by the amount F .

- Discontinuities in load = Jumps in Shear.



3. Bending Moment Diagram (M_x)



Notice that the above shear force diagram is the derivative of this bending moment diagram!

Shows the bending effort. This is the most important diagram for failure.

- **Key Insight:** The moment is Zero at free ends (A and B).
- **The Peak:** It peaks under the load (Point C). This is where the beam snaps ($M_{max} = -2Fa/3$).

Properties of Diagrams

The Power of Derivatives

You do not need to cut the beam 10 times. The shape of the diagrams is mathematically predicted by the loads.

- Load (q) determines the Slope of Shear (T).
- Shear (T) determines the Slope of Moment (M).

A. The Differential Rules

These rules apply to every infinitesimal slice of the beam (dz).

1. Slope of Shear = -Load

$$\frac{dT}{dz} = -q(z)$$

- **Translation:** The value of the distributed load q tells you how steep the Shear diagram is.
- **Example:** If $q = 0$ (no load), the slope is 0 (Shear is horizontal).

2. Slope of Moment = Shear

$$\frac{dM}{dz} = T(z)$$

- **Translation:** The value of the Shear T tells you how steep the Moment diagram is.
- **The "Order" Rule:** The mathematical order increases by 1 each step.
 - Load: Constant \rightarrow Shear: Linear \rightarrow Moment: Parabolic.

B. The "Graphical Method" (Sketching)

Use these relationships to spot errors or sketch rapidly.

Feature	Effect on Shear (T)	Effect on Moment (M)
No Load ($q = 0$)	Constant (Horizontal Line)	Linear (Slope = T)
Uniform Load ($q = c$)	Linear (Sloped Line)	Parabolic (Curve)
Point Force (F)	JUMP (Vertical shift by F)	Kink (Change in slope)
Couple (C)	No change	JUMP (Vertical shift by C)

C. Finding the Critical Point (Max Moment)

⚡ Critical Rule

Since $\frac{dM}{dz} = T$, the Maximum (or Minimum) Bending Moment occurs exactly where the Shear Force is ZERO.

$$T(z) = 0 \implies M_{max}$$

Strategy: Do not calculate moment everywhere. Find where T crosses the axis, and calculate M only at that specific point.

4. Stress

4. Stress

Stress Vector

$$\vec{t} = \frac{d\vec{F}_n}{dA} = \begin{bmatrix} \sigma_{nn} \\ \tau_{nm} \\ \tau_{np} \end{bmatrix}$$

Where σ is the normal stress and τ is tangential stress. The coordinate space is normal-space.

Stress components have two indices (subscripts), σ_{ii}, τ_{ik} :

- The first is the axis of the plane cutting the area
- The second is the direction of the force

Stress Tensor

⚡ Reciprocity of stress

$$\tau_{xy} = \tau_{yx} \quad \tau_{xz} = \tau_{zx} \quad \tau_{yz} = \tau_{zy}$$

Therefore, we only have 6 total stress components in an infinitesimal volume.
(Including $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$)

The above is true in statics, because if it wasn't then we would get movement since net forces would not be zero.

$$\vec{t} = [\sigma]\vec{n}$$

Where:

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Principal Stresses

A principal stress λ is one such that:

$$\vec{t} = [\sigma]\vec{n} = \lambda\vec{n}$$

Recalling from [Linear Algebra and Geometry](#), we can notice that principal stresses are eigenvalues of the stress tensor. We can therefore find them by using the formula:

$$\det([\sigma] - \lambda I) = 0$$

Where I is the identity matrix. The associated eigenvectors are the **principal directions**.

In the principal frame of reference, the stress tensor becomes:

$$[\sigma] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Where $\sigma_1, \sigma_2, \sigma_3$ are principal stresses (previously λ).



Note



For all stress tensors, principal stresses exist, since $[\sigma]$ is symmetrical.

We can use [Mohr's Circle](#) to calculate components of the stress tensor.

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}}_{\substack{\text{Isotropic} \\ \text{(Volume changes)}}} + \underbrace{\begin{bmatrix} \sigma_{xx} - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} - \sigma_m & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} - \sigma_m \end{bmatrix}}_{\substack{\text{Deviatoric} \\ \text{(Shape changes)}}$$

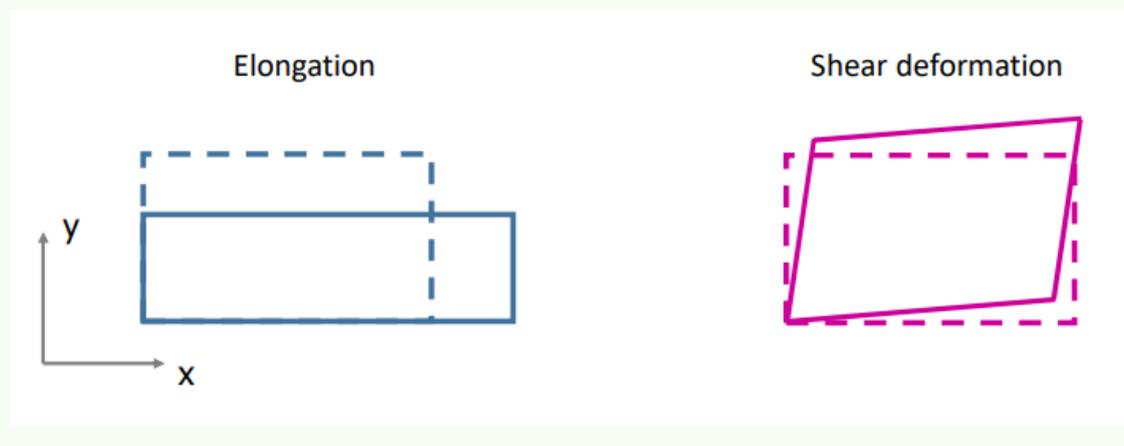
$$\sigma_m = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} \quad (\text{Hydrostatic Stress})$$

5. Strain

5. Strain

Strain is deformation caused by an applied [stress](#).

✓ Types of Strain



- **Elongation Strain (ϵ):** It measures how much a material stretches or squashes in a specific direction (change in length per unit length).
- **Shear Strain (γ):** It measures how much the angle between two originally perpendicular lines changes (distortion of the corner of a square).

$$\epsilon = \frac{\partial u}{\partial x}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

The Strain Tensor

Just like stress, strain is a tensor field. It varies point-by-point.

$$[\epsilon] = \begin{bmatrix} \epsilon_x & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_y & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_z \end{bmatrix} = \begin{bmatrix} \epsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \epsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \epsilon_z \end{bmatrix}$$

⚡ Trap: Engineering vs. Tensor Shear

The tensor components (ε_{ij}) are **NOT** the same as the "Engineering Shear Strain" (γ_{xy}) used in calculations.

- **Tensor Shear:** $\varepsilon_{xy} = \frac{1}{2}\gamma_{xy}$
- **Engineering Shear:** $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

Why? The tensor must be symmetric. The factor of 1/2 splits the total angle change (γ) equally between the two faces.

Volumetric Strain (ε_v)

The sum of the diagonal terms (the Trace) tells you how much the **Volume** changes (Dilation).

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z = \text{Trace}([\varepsilon]) = \frac{\Delta V}{V}$$

- $\varepsilon_v > 0$: Expansion (Dilation).
- $\varepsilon_v < 0$: Contraction (Compaction).
- $\varepsilon_v = 0$: Incompressible material (Volume is constant, only shape changes).

Principal Strains

Exactly like Stress, there is a specific orientation where **Shear Strain is zero**.

- These are the **Principal Strains** ($\varepsilon_1, \varepsilon_2, \varepsilon_3$).
- You can find them using [Mohr's Circle for strain](#) (identical process to stress, just plot ε on x-axis and $\gamma/2$ on y-axis).

The Displacement Gradient (**J**)

The derivative of displacement ($\nabla \mathbf{u}$) tells us how a point moves relative to its neighbours. It contains two things mixed together: **Deformation and Rigid Rotation**.

We mathematically split it to isolate the strain:

$$\mathbf{J} = \nabla \mathbf{u} = \underbrace{\boldsymbol{\varepsilon}}_{\text{Strain}} + \underbrace{\boldsymbol{\Omega}}_{\text{Rotation}}$$

1. The Strain Tensor ($\boldsymbol{\varepsilon}$)

- **Maths:** The Symmetric part of **J**.

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{J} + \mathbf{J}^T)$$

- **Physics:** Represents pure stretching and shape change. **Causes Stress.**

2. The Spin Tensor (Ω)

- **Maths:** The **Skew-Symmetric** part of **J**.

$$\Omega = \frac{1}{2}(\mathbf{J} - \mathbf{J}^T)$$

- **Physics:** Represents rigid body rotation. **Causes NO Stress.**

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{bmatrix} = \underbrace{\begin{bmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{bmatrix}}_{\substack{\text{Isotropic} \\ \text{(Volume Changes)}}} + \underbrace{\begin{bmatrix} \varepsilon_x - \varepsilon_m & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y - \varepsilon_m & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z - \varepsilon_m \end{bmatrix}}_{\substack{\text{Deviatoric} \\ \text{(Shape Changes)}}$$

$$\varepsilon_m = \frac{\varepsilon_x + \varepsilon_y + \varepsilon_z}{3} = \frac{1}{3}\varepsilon_v \quad (\text{Hydrostatic Strain})$$

Constitutive Equations (Hooke's Law)

Linear Elasticity (1D)

For a simple bar pulled in one direction (Uniaxial):

Hooke's Law

$$\sigma = E\varepsilon$$

- E : Young's Modulus (Stiffness). Higher E = Stiffer material.

We can use Hooke's law to get force and work done in two occasions:

$$F = k\Delta x$$

1. Instantaneous (Constant Force)

Force F is at 100% from start to finish.

- $W = F\Delta x = k\Delta x^2$
- *Note:* Contains 2x the energy of the progressive case.

2. Progressive (Linear Force)

Force F grows from 0 to max as displacement increases.

- $W = \int_0^{\Delta x} k\tau d\tau = \frac{1}{2}k\Delta x^2$
- *Note:* Area of the triangle under the $F/\Delta x$ curve ($W = \frac{1}{2}F\Delta x$).

Poisson's Ratio (ν)

When you pull a rubber band, it gets thinner. Poisson's ratio measures this necking effect.

$$\nu = -\frac{\varepsilon_{transverse}}{\varepsilon_{longitudinal}}$$

- Range: $0 \leq \nu \leq 0.5$ (Most metals ≈ 0.3).

Energy Density

$$\eta_{\text{normal}} = \frac{1}{2} \sigma \varepsilon$$

$$\eta_{\text{shear}} = \frac{1}{2} \tau \gamma$$

$$\eta_{\text{tot}} = \frac{1}{2} (\sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + \sigma_{zz} \varepsilon_{zz} + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yz} \gamma_{yz})$$

Generalized Hooke's Law (3D)

In the real world, stress in one direction (x) causes strain in *all* directions (due to Poisson). For an **Isotropic** material (same properties in all directions), the equations are:

Normal Strains

$$\begin{cases} \varepsilon_x = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \varepsilon_y = \frac{1}{E} [\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \varepsilon_z = \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] \end{cases}$$

Shear Strains

Shear is decoupled from normal stress. It depends on the **Shear Modulus (G)**.

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \quad \gamma_{yz} = \frac{\tau_{yz}}{G} \quad \gamma_{xz} = \frac{\tau_{xz}}{G}$$

Shear Modulus

$$G = \frac{E}{2(1 + \nu)}$$

In matrix form:

$$\vec{\sigma} = [C] \vec{\varepsilon}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix}}_{\text{C (Stiffness Matrix)}} \cdot \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}$$

- For **Isotropic** materials (standard metal), this giant matrix is mostly zeros. You only need two numbers to fill it: E and ν .
- The matrix is **Symmetric** ($C_{ij} = C_{ji}$), so you "only" need 21 constants for the worst-case scenario (like wood or composites), not 36.

Thermal Strain

Temperature changes cause expansion/contraction **without** stress (unless constrained).

$$\varepsilon_T = \alpha \cdot \Delta T$$

- α : Coefficient of thermal expansion.
- **Total Strain**: The sum of mechanical stress + thermal expansion.

$$\varepsilon_{total} = \varepsilon_{mech} + \varepsilon_{th} = \frac{\sigma}{E} + \alpha \Delta T$$

Some Properties

Metal	E/Nmm^{-2}	ν
Steel	2×10^5	0.3
Cast Iron	1×10^5 to 1.8×10^5	0.27
<i>Ti</i>	1.2×10^5	0.3
<i>Al</i>	7×10^4	0.3

$\sigma_{all} \approx 1000 Nmm^{-2}$ (max allowed stress) for steel, which corresponds to $\varepsilon = 0.005$.

Material (minimum values)	Type/Grade	σ_y (MPa)	σ_{ult} (MPa)	A%
STEEL - Structural	S 235	235	360	26
(UNI EN 10025)	S 275	275	430	22
	S 355	355	510	22
STEEL - annealed	C 30	400	600	18
(UNI EN 10083)	C 60	580	850	11
	41Cr4	800	1000	11
	36NiCrMo3	1050	1250	9
CAST IRON	G10	-	100	-
Gray	G20	-	200	-
	G30	-	290	-
CAST IRON	Gs370-17	230	370	17
Spheroidal	Gs500-7	320	500	7

Material (minimum values)	Type/Grade	σ_y (MPa)	σ_{ult} (MPa)	A%
	Gs700-2	420	700	2

 Note

Values represent minimum thresholds. For Gray Cast Iron, yield strength (σ_y) and elongation (A%) are typically not defined due to the material's brittle nature.

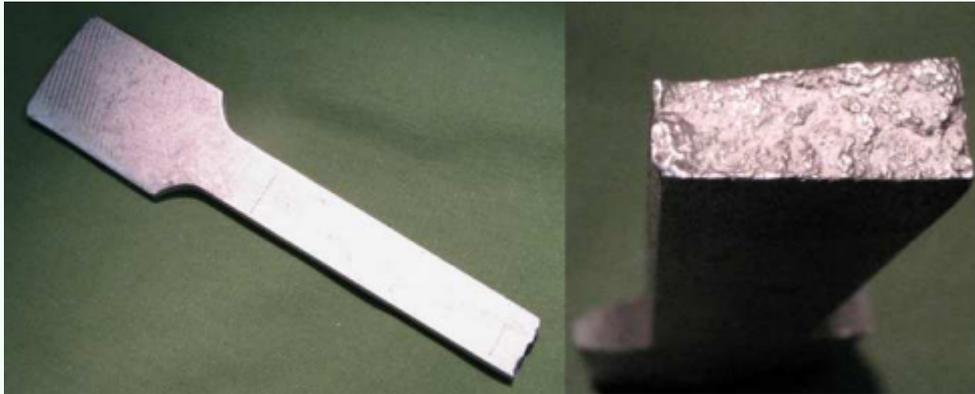
6. Failure

6. Failure

Tip

Ductile materials fracture after necking (yielding), while brittle materials will suddenly snap with an uneven pattern (snapping).

Brittle:



Ductile:



Von Mises Criterion

Not to be confused with the [Taylor-Von Mises Criterion](#) from STM/TMM!

Summary

The von Mises criterion states that a ductile material yields when the distortion (deviatoric) energy density at a point reaches the critical value observed at the yield limit in a simple uniaxial tensile test.

Ductile materials (like steel) do not care if you squeeze them evenly from all sides (pressure). They only care if you try to **twist or warp** them (distortion).

The Energy Split

We can separate the total energy inside the material into two buckets:

- **Bucket A (Safe):** Energy that changes **volume**. This is caused by “Hydrostatic Stress” (pressure). It does *not* cause yielding.
- **Bucket B (Dangerous):** Energy that changes **shape** (distortion/warping). This is caused by "Deviatoric Stress". This is what breaks the material.

The Result: “Equivalent Stress”

The derivation boils all that complex 3D stress down to a **single number** called the **von Mises Equivalent Stress** (σ_{eq} or σ_{VM}).

- **Why?** So you can compare a complex real-world mess (3D stress) to a simple 1D [Tensile Test](#).
- **The Rule:** If your calculated “Distortion Number” (σ_{eq}) is higher than the Yield Strength (S_y) from the simple test, the part fails.

The Only Formula You Actually Need

Forget the derivation. This is the weapon you wield:

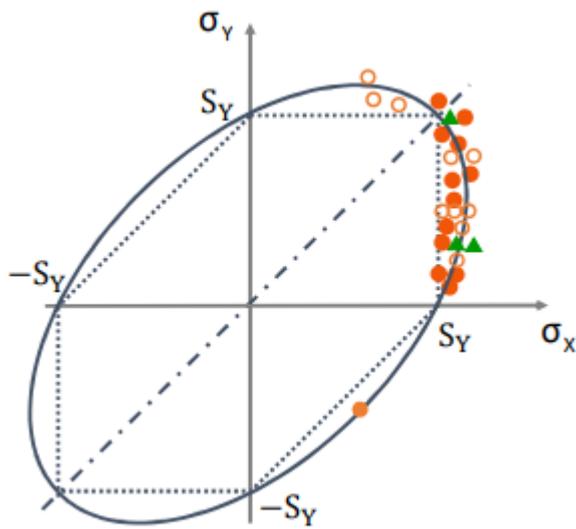
$$\sigma_{eq} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}$$

If $\sigma_{eq} > S_y$, it yields.

Comparison: Tresca vs. Von Mises

Visualizing the Yield Surface

- **Tresca (Hexagon):** The inner shape. Conservative (predicts failure earlier). It "cuts the corners".
- **Von Mises (Ellipse):** The outer shape. It allows higher stress states in the corners (shear + tension).
- **Reality:** Experimental data (orange dots) clings to the **Von Mises Ellipse**, making it the accurate choice for ductile metals.



Summary Checklist

Material	Criterion	Formula (σ_{eq})	Limit
Ductile	Tresca	$\sigma_1 - \sigma_3$	S_y
Ductile	Von Mises	$\frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}$	S_y
Brittle	Rankine	σ_1 (Max Principal)	S_u

Safety Factor (F_S)

The Law: Real components are almost never designed to the limit. We apply a margin for uncertainty (loads, material defects).

$$F_S = \frac{\text{Limit Stress}}{\text{Equivalent Stress}} > 1$$

- **Brittle Materials:** High Safety Factor ($F_S \geq 3$). Failure is sudden and catastrophic.
- **Ductile Materials:** Lower Safety Factor ($F_S \geq 1.5$). Material yields (warns you) before snapping.

Allowable Stress (σ_{adm})

Sometimes standards give you the max allowed stress directly instead of an F_S :

- $\sigma_{eq} \leq \sigma_{adm}$ where $\sigma_{adm} = \frac{S_{limit}}{F_S}$.

7. Beam Theory

7. Beam Theory

Summary

Beam theory describes what happens between a load and stress by analysing internal actions. It is used in the following analysis workflow:

1. Loads
2. Internal Actions
3. Stress
4. Strain
5. Displacements and rotations

De Saint Venant Theory

Assumptions

- The solid analysed is a **beam** (prismatic, rectilinear axis z).
- **External Loading:** No forces applied to the lateral surfaces (mantle); loads act only at the ends (bases).
- **Internal Stress State:** The stress components in the plane of the cross-section are negligible:

$$\sigma_{xx} = \sigma_{yy} = \tau_{xy} = 0$$

Stress Resultants (Internal Forces)

$$\underbrace{N = \int_A \sigma_{zz} dA}_{\text{Normal force}} \quad \underbrace{T_x = \int_A \tau_{xz} dA}_{\text{Shear force (x)}} \quad \underbrace{T_y = \int_A \tau_{yz} dA}_{\text{Shear force (y)}}$$

$$\underbrace{M_x = \int_A \sigma_{zz} y dA}_{\text{Bending moment (x)}} \quad \underbrace{M_y = - \int_A \sigma_{zz} x dA}_{\text{Bending moment (y)}} \quad \underbrace{M_z = \int_A (\tau_{yz} x - \tau_{xz} y) dA}_{\text{Torque}}$$

Superposition of Effects

Tip

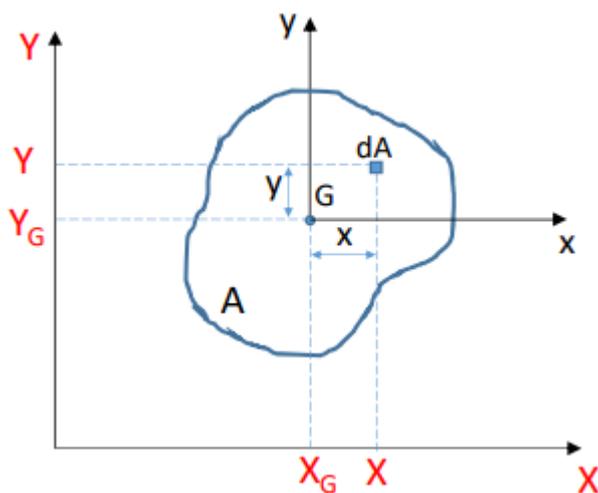
Because we assume the material behaves with **Linear Elasticity**, the **Principle of Superposition** applies.

You do not need to solve for all stresses simultaneously.

1. Calculate the stress for each internal action (N , M_x , M_y , T_x , T_y , M_z) **separately**.
2. Sum the results algebraically to find the total stress at any point.

$$\sigma_{total} = \sigma_N + \sigma_{M_x} + \sigma_{M_y} + \dots$$

Geometry of Areas



First Moment of Area (S or Q)

Measure of area distribution relative to an axis.

$$S_x = Q_x = \int_A y dA \quad ; \quad S_y = Q_y = \int_A x dA$$

Centroid (G)

The point (X_G , Y_G) about which the first moment of area is zero.

Property:

$$\int_A (x - X_G) dA = 0 \quad ; \quad \int_A (y - Y_G) dA = 0$$

Coordinates:

$$X_G = \frac{S_y}{A} = \frac{\int x dA}{\int dA} \quad ; \quad Y_G = \frac{S_x}{A} = \frac{\int y dA}{\int dA}$$

Centroid of Composite Areas

For complex shapes made of simple geometric parts (e.g., I-beams, T-sections).

Principle:

Replace integration with the summation of individual parts.

$$X_G = \frac{\sum_i A_i X_{Gi}}{\sum_i A_i} \quad ; \quad Y_G = \frac{\sum_i A_i Y_{Gi}}{\sum_i A_i}$$

- A_i : Area of the i -th simple shape.
- X_{Gi}, Y_{Gi} : Centroid coordinates of the i -th simple shape.

Second Moment of Area (Area Moment of Inertia)

Measures the distribution of area relative to an axis (resistance to bending).

Definitions:

$$I_{xx} = \int_A y^2 dA \quad (\text{About x-axis})$$

$$I_{yy} = \int_A x^2 dA \quad (\text{About y-axis})$$

$$I_{xy} = \int_A xy dA \quad (\text{Mixed / Centrifugal Moment})$$

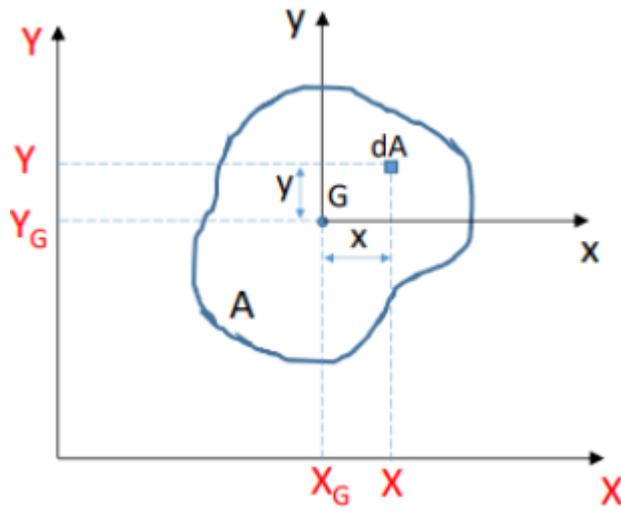
$$I_{tot} = \Sigma I$$

Terminology

If computed about the centroidal axes (x, y passing through G), they are called Central (Centroidal) Moments of Inertia.

Transportation Theorem (Huygens-Steiner)

Relates moments of inertia about centroidal axes to any parallel axes.



Formula:

$$I_{XX} = I_{xx} + AY_G^2$$

$$I_{YY} = I_{yy} + AX_G^2$$

$$I_{XY} = I_{xy} + AX_GY_G$$

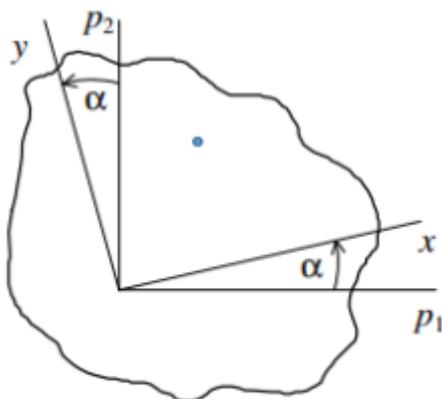
- I_{xx}, I_{yy}, I_{xy} : Central (centroidal) moments of inertia.
- I_{XX}, I_{YY}, I_{XY} : Moments of inertia about the new parallel axes X, Y .
- X_G, Y_G : Coordinates of the centroid in the new X, Y system.
- A : Total Area.

Tip

The moment of inertia is always minimum about the centroidal axis. Moving the axis away always *increases* the inertia by Ad^2 .

Principal Moments of Inertia

The specific rotation of axes for which the Centrifugal Moment (Product of Inertia) vanishes ($I_{xy} = 0$).



α : angle between p_1 and x axes

- **Principal Axes (1, 2):** The axes about which I is Maximum (I_1) and Minimum (I_2).
- **Property:** These axes are always orthogonal (90° apart). Symmetry axes are always principal axes.

Formulas for Principal Moments:

Given I_{xx}, I_{yy}, I_{xy} in an arbitrary system, the principal values are:

$$I_{1,2} = \frac{I_{xx} + I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$$

Orientation of Principal Axes

The angle α required to rotate the axes to the principal orientation.

$$|\tan(2\alpha)| = \left| \frac{2I_{xy}}{I_{xx} - I_{yy}} \right|$$

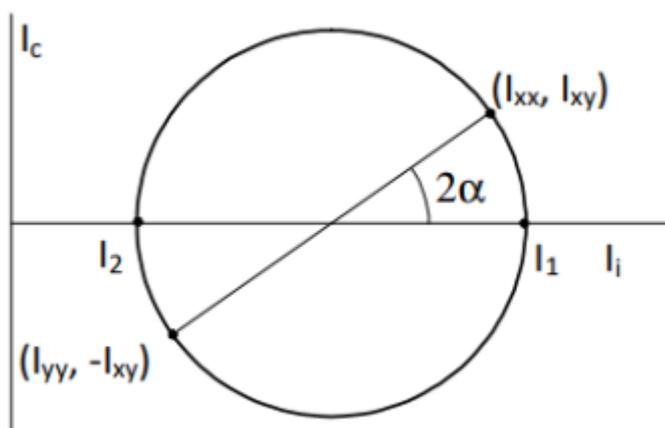
⚠ Determining the Quadrant

The value of 2α depends on the signs of the numerator and denominator:

- If $I_{xx} \geq I_{yy}$ and $I_{xy} \geq 0 \rightarrow 0^\circ \leq 2\alpha \leq 90^\circ$
- If $I_{xx} \leq I_{yy}$ and $I_{xy} \geq 0 \rightarrow 90^\circ \leq 2\alpha \leq 180^\circ$

Mohr's Circle for Inertia

We can use [Mohr's Circle](#) to represent the variation of moments of inertia with respect to axis rotation.



- **Centre:** $C = \left(\frac{I_{xx} + I_{yy}}{2}, 0\right)$
- **Radius:** $R = \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$

Axial Force and Normal Stress

The Kinematic Assumption

Consider a beam subjected *only* to [axial](#) elongation.

- **Hypothesis:** Cross-sections remain plane and parallel to each other. They move axially (dw) but do not rotate.
- **Consequence:** The elongation dw is uniform over the cross-section.

Stress Distribution

Since the strain is constant ($\varepsilon_{zz} = \frac{dw}{dz} = \text{const}$), the material law (Hooke's Law) implies the stress is also constant.

$$\sigma_{zz} = E \cdot \varepsilon_{zz} = \text{constant}$$

Integrating this constant stress over the area yields the fundamental formula:

$$\sigma_{zz} = \frac{N}{A}$$

- N : Internal Normal Force.
- A : Area of the cross-section.

The Centroid Condition

IMPORTANT

The above formula is **only valid** if the external force is applied at the **Centroid (G)** of the cross-section.

Proof:

If σ_{zz} is constant, the bending moments generated are:

$$M_x = \int_A \sigma_{zz} y dA = \sigma_{zz} \int_A y dA = \sigma_{zz} S_x$$

$$M_y = \int_A -\sigma_{zz} x dA = -\sigma_{zz} \int_A x dA = -\sigma_{zz} S_y$$

For M_x and M_y to be zero (pure axial force), the static moments S_x and S_y must be zero. This is only true if the origin is the **Centroid**.

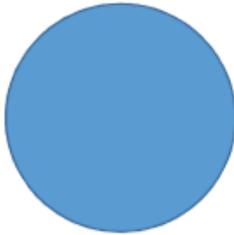
⚠ Eccentric Loads

If the load N is applied away from the centroid (eccentricity), it generates **Bending Moments** ($M = N \cdot e$) and you cannot use $\sigma = N/A$ alone. You must add the bending stress terms.

Example Calculation

Example

Circular cross section

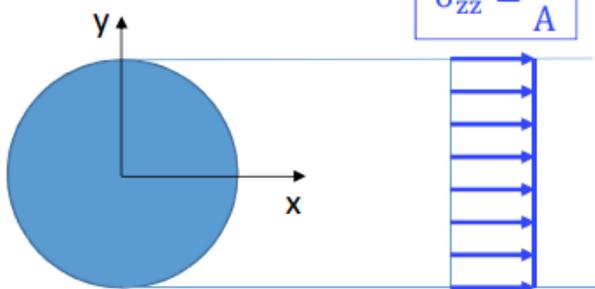


$D = 30 \text{ mm}$ (diameter)

$N = 10 \text{ kN}$ (axial load)

$$\sigma = \frac{N}{A} = \frac{4N}{\pi D^2} = \frac{4 \cdot 10000}{900\pi} = 14 \text{ MPa}$$

Distribution of normal stress



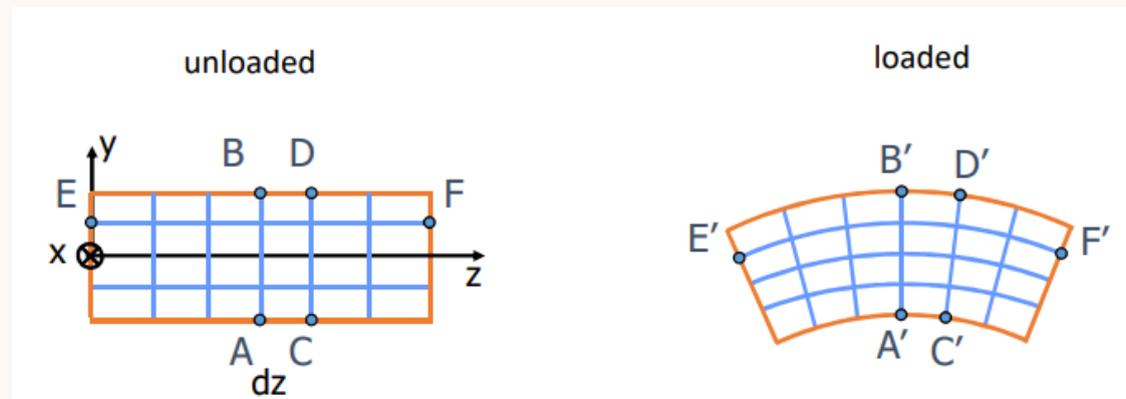
12.6 Flexural Behaviour (Bending)

⚠ Kinematic Assumptions (Bernoulli-Navier)

Consider a beam subjected to Bending Moments (M_x, M_y).

- **Hypothesis:** Cross-sections remain plane and perpendicular to the deformed longitudinal axis.
- **Consequence:** The strain ϵ_{zz} varies linearly over the cross-section.

$$\epsilon_{zz} = k_x y \quad \text{or} \quad \epsilon_{zz} = -k_y x$$



The Navier Formulas (Stress Distribution)

Assuming the coordinate system is **Centroidal** (origin at G) and **Principal** ($I_{xy} = 0$), the relationship between moment and stress is linear.

1. Bending about X-axis (M_x):

$$\sigma_{zz} = \frac{M_x}{I_{xx}} y$$

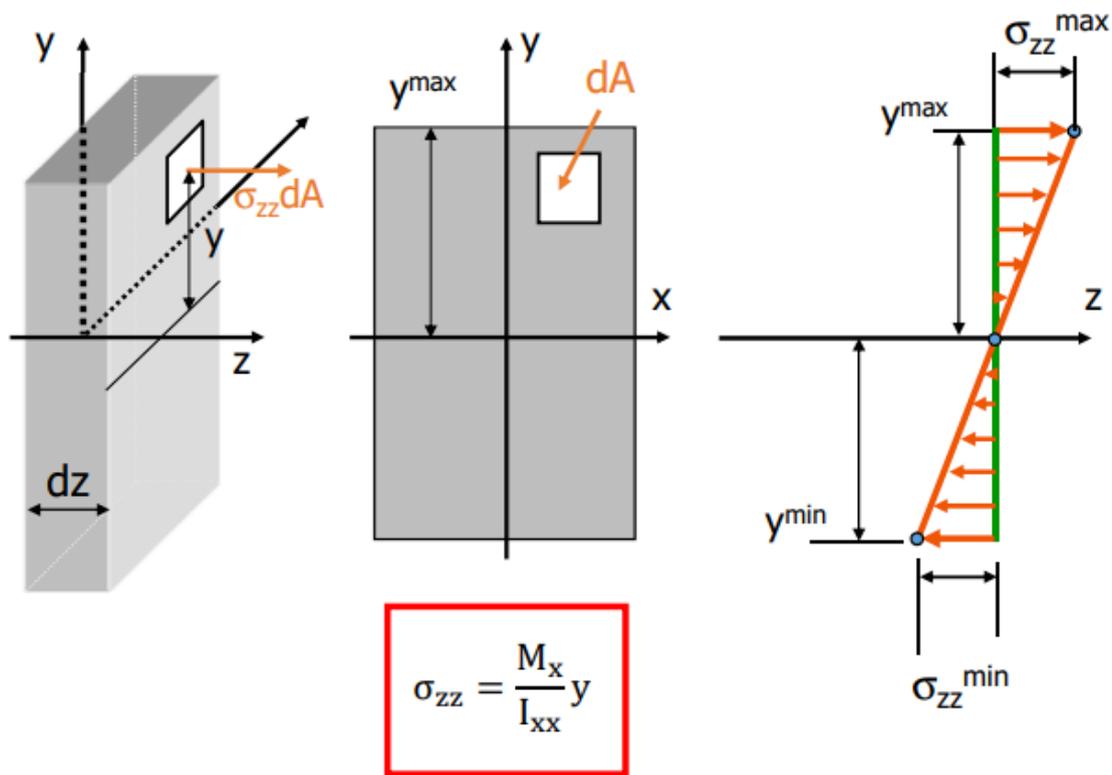
2. Bending about Y-axis (M_y):

$$\sigma_{zz} = -\frac{M_y}{I_{yy}} x$$

The Neutral Axis

The axis where the normal stress is zero ($\sigma_{zz} = 0$).

- In pure bending, the neutral axis passes through the **Centroid**.
- It divides the section into a region of **Tension** ($\sigma > 0$) and **Compression** ($\sigma < 0$).



The maximum stress occurs at the point furthest from the neutral axis (y_{max}).

$$\sigma_{max} = \frac{M_x}{I_{xx}} y_{max}$$

The Elastic Line (Beam Deflection)

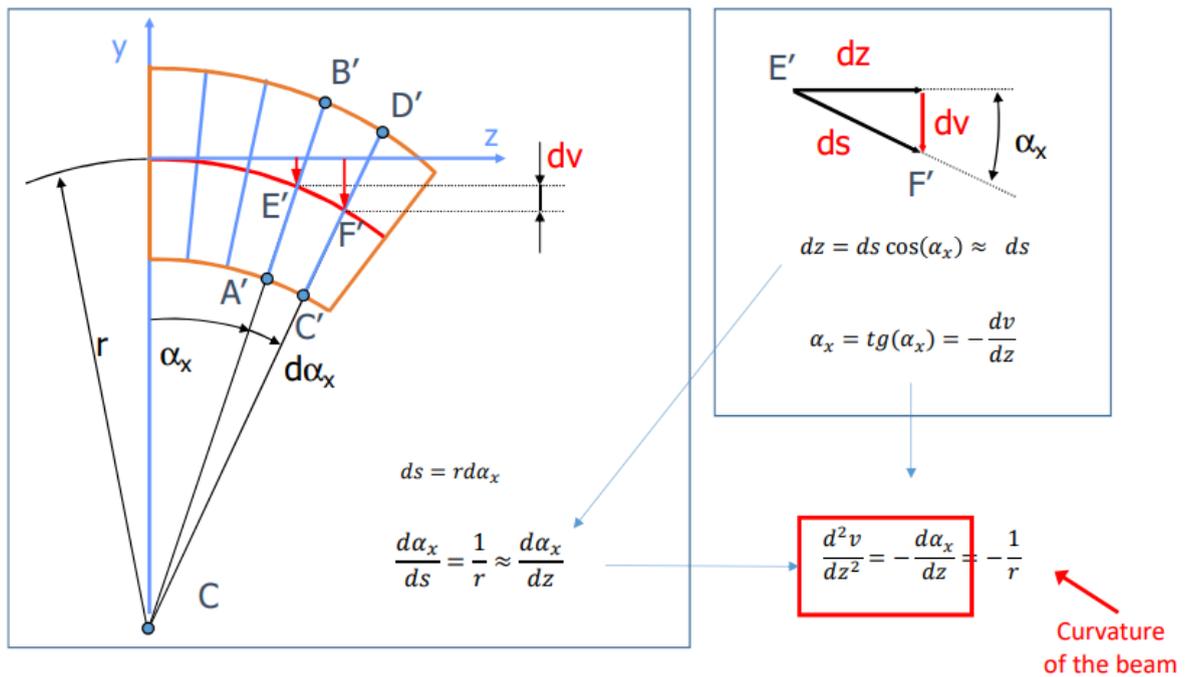
We relate the internal bending moment to the physical deformation (curvature) of the beam.

Differential Equation of the Elastic Line

For small deformations, the curvature $\chi \approx v''(z)$. The fundamental equation governing beam deflection is:

$$\frac{d^2 v}{dz^2} = -\frac{M_x(z)}{EI_{xx}}$$

- $v(z)$: Vertical displacement (deflection) of the beam.
- EI_{xx} : Flexural Stiffness (resistance to bending).
- $M_x(z)$: Bending moment function along the beam axis z .



Solving for Deflection (Integration Method)

To find the deformed shape $v(z)$, you must integrate the moment equation twice.

1. Integrate once (Slope/Rotation α_x):

$$\alpha_x(z) = \frac{dv}{dz} = \int -\frac{M_x(z)}{EI_{xx}} dz + C_1$$

2. Integrate twice (Deflection v):

$$v(z) = \int \alpha_x(z) dz = \iint -\frac{M_x(z)}{EI_{xx}} dz dz + C_1 z + C_2$$

Boundary Conditions:

The constants C_1 and C_2 are determined by the constraints (supports):

- **Clamp/Fixed Support:** $v = 0, \alpha = 0$.
- **Hinge/Pin Support:** $v = 0$.
- **Symmetry:** At a plane of symmetry, the rotation is zero ($\alpha = 0$).

🔄 Exploit Symmetry

If the beam geometry and loading are symmetrical, you only need to solve half the beam.

Set the rotation to zero at the symmetry point: $\alpha(L/2) = 0$. This simplifies the algebra significantly.

Skew Bending (Bending about a Non-Principal Axis)

When the total bending moment vector M_f is not parallel to one of the principal axes (x, y), it must be decomposed into its components.

Stress Calculation (Superposition)

Due to linearity, we sum the effects of bending about x and y simultaneously.

$$\sigma_{zz} = \frac{M_x}{I_{xx}}y - \frac{M_y}{I_{yy}}x$$

- M_x, M_y : Components of the bending moment ($M_x = M_f \cos \theta, M_y = M_f \sin \theta$).
- I_{xx}, I_{yy} : Principal moments of inertia.
- x, y : Coordinates of the point where stress is calculated.

The Neutral Axis (n.a.)

The line where normal stress is zero ($\sigma_{zz} = 0$).

Orientation:

The angle of the neutral axis (ϕ) is **not** usually perpendicular to the moment vector (θ). It depends on the ratio of stiffnesses.

$$\tan(\phi) = \frac{y}{x} = \frac{M_y \cdot I_{xx}}{M_x \cdot I_{yy}}$$

Physical Meaning

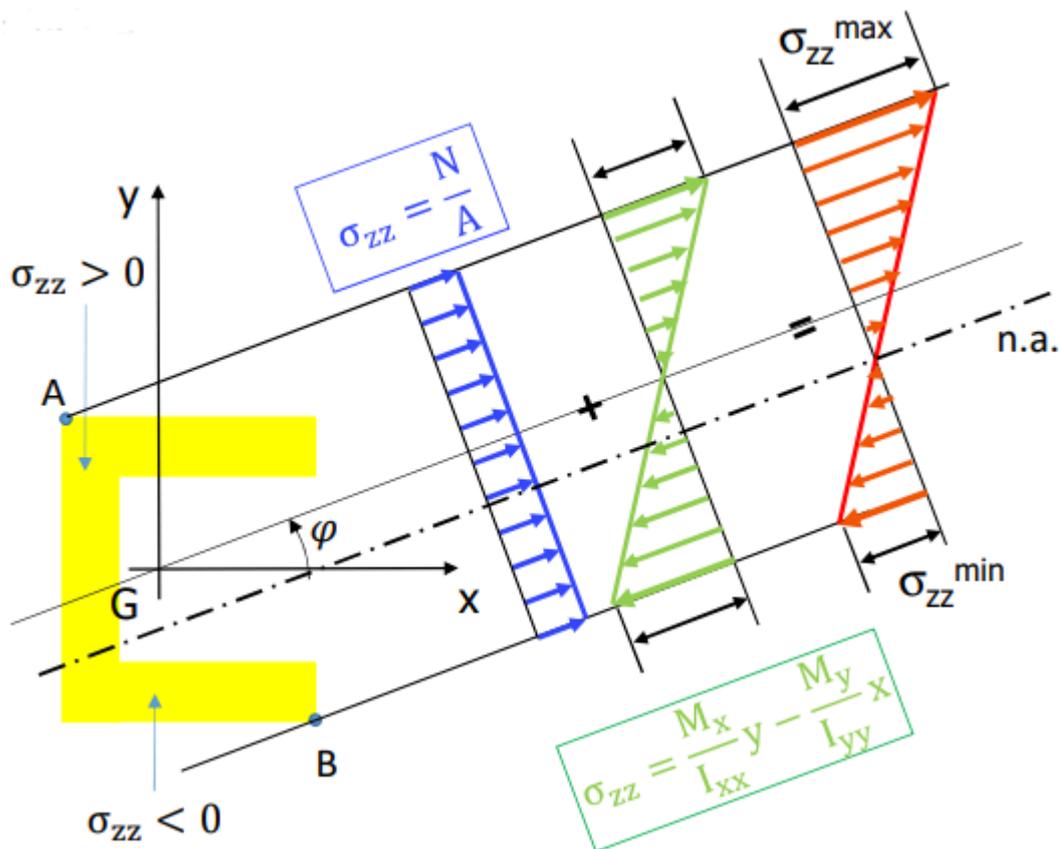
Because beams are often stiffer in one direction than the other ($I_{xx} \neq I_{yy}$), the beam tends to bend "towards" the weaker axis, causing the neutral axis to rotate away from the moment vector.

Graphical Procedure for Stress Distribution

To draw the stress diagram correctly:

1. **Find the Neutral Axis:** Calculate ϕ and draw the n.a. passing through the centroid G .
2. **Identify Critical Points:** Find the points on the cross-section furthest from the n.a. (usually corners A and B). These are the locations of σ_{max} and σ_{min} .
3. **Draw Perpendiculars:** Project these points onto a line perpendicular to the n.a.
4. **Plot Stress:** Draw the linear variation from maximum tension to maximum

compression.



When a beam carries both an Axial Force (N) and Bending Moments (M_x, M_y), the total normal stress is the algebraic sum of the individual effects.

The General Formula (Navier)

$$\sigma_{zz} = \underbrace{\frac{N}{A}}_{\text{Axial}} + \underbrace{\frac{M_x}{I_{xx}}y - \frac{M_y}{I_{yy}}x}_{\text{Bending}}$$

The Shifted Neutral Axis

Unlike in pure bending, the Neutral Axis (*n. a.*) ($\sigma_{zz} = 0$) does not pass through the centroid when an axial force N is present.

- If N is tension (+), the n.a. shifts away from the tension side.
- If N is compression (−), the n.a. shifts away from the compression side.

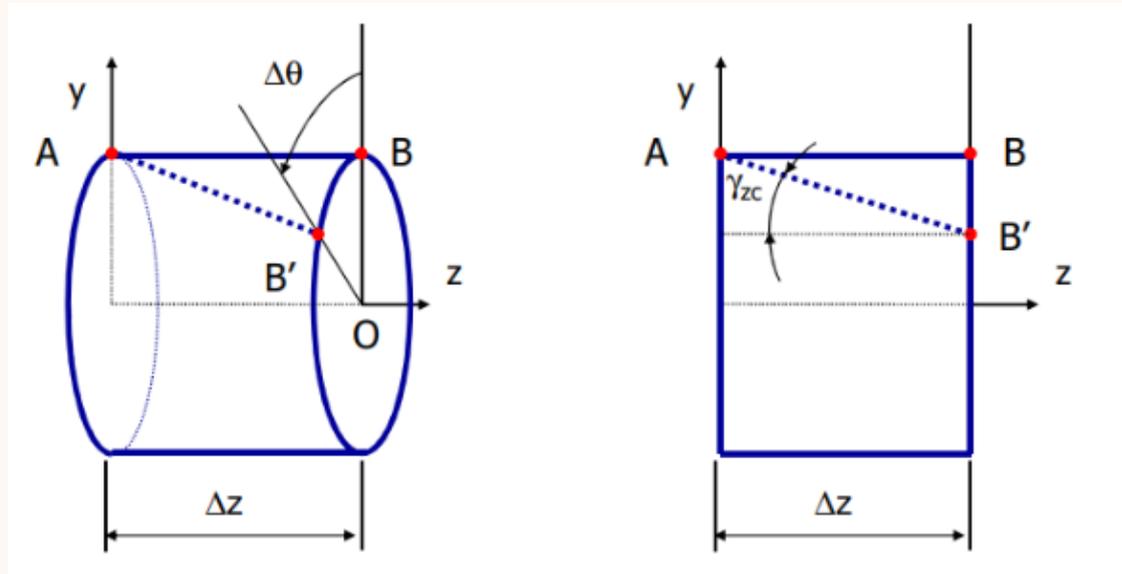
Torsional Behaviour (Circular Sections)

Applies strictly to Circular (solid or hollow) cross-sections.

- **Note:** For non-circular sections (e.g., rectangles, I-beams), the cross-section warps (does not remain flat), and these formulas do not apply.

⚠ Kinematic Assumptions

- **Rigid Rotation:** Cross-sections remain plane and circular. They rotate rigidly by an angle $\Delta\theta$ around the z-axis.
- **Straight Radii:** Diameters remain straight lines.



Strain and Stress Distribution

The shear strain γ_{zc} varies linearly with the distance r from the centre.

$$\gamma_{zc} = \frac{d\theta}{dz} r$$

Applying Hooke's Law ($\tau = G\gamma$), the shear stress τ_{zc} is also linear:

$$\tau_{zc} = G \frac{d\theta}{dz} r$$

The Torsion Formula

Relating the internal Torque (M_z) to the twist and stress.

Angle of Twist (Stiffness):

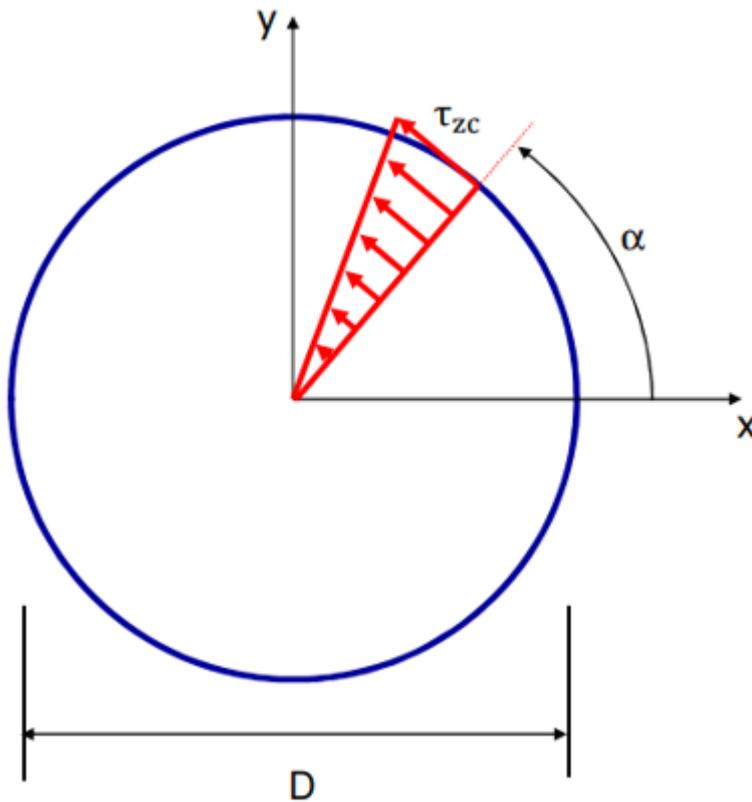
$$\frac{d\theta}{dz} = \frac{M_z}{GI_p}$$

Shear Stress Distribution:

$$\tau_{zc} = \frac{M_z}{I_p} r$$

- M_z : Applied Torque.
- G : Shear Modulus of the material.
- r : Radial distance from centre ($0 \leq r \leq D/2$).

- I_p : Polar Moment of Inertia.



Polar Moment of Inertia (I_p)

The geometric resistance to torsion. For circular sections, $I_p = I_{xx} + I_{yy}$.

Section Type	Formula for I_p	Diagram
Solid Circle	$I_p = \frac{\pi D^4}{32}$	
Hollow Circle	$I_p = \frac{\pi(D^4 - d^4)}{32}$	

Maximum Stress

The maximum stress τ_{max} always occurs at the outer surface ($r = D/2$).

$$\tau_{max} = \frac{M_z}{I_p} \frac{D}{2}$$

Torsion of Thin-Walled Open Sections

When the cross-section is not circular (e.g., a rectangle or an I-beam), the cross-section **warps** (bulges in the z-direction), and the simple I_p formulas no longer apply.

Single Rectangular Section

For a rectangle of width a and thickness b (where $a > b$):

- **Maximum Shear Stress:** Occurs at the middle of the long edge.

$$\tau_{max} = \frac{M_z}{C_1 ab^2}$$

- **Angle of Twist:**

$$\frac{d\theta}{dz} = \frac{M_z}{C_2 Gab^3}$$

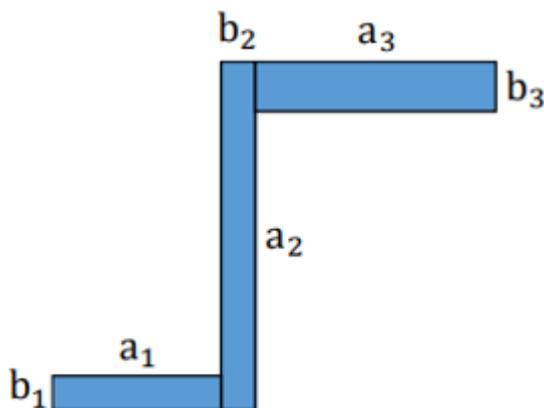
Coefficients C_1 and C_2

These depend on the ratio a/b .

- For thin sections ($a/b \geq 10$), $C_1 = C_2 \approx 1/3$ (or 0.333).

Composite Open Sections (L, T, I shapes)

We approximate complex open shapes as a collection of N individual rectangles.



1. **Equivalent Stiffness ($I_{p,eq}$):** Sum the stiffness of individual rectangles.

$$I_{p,eq} \approx \sum_{i=1}^N \frac{1}{3} a_i b_i^3$$

(Note: This assumes thin walls where $C_2 \approx 1/3$)

2. **Angle of Twist:** The entire section rotates together.

$$\frac{d\theta}{dz} = \frac{M_z}{GI_{p,eq}}$$

3. **Max Shear Stress:**

The stress is highest in the **thickest** part of the section (b_{max}).

$$\tau_{max,i} = \frac{M_z}{I_{p,eq}} b_i$$

Torsion of Thin-Walled Closed Sections (Bredt's Theory)

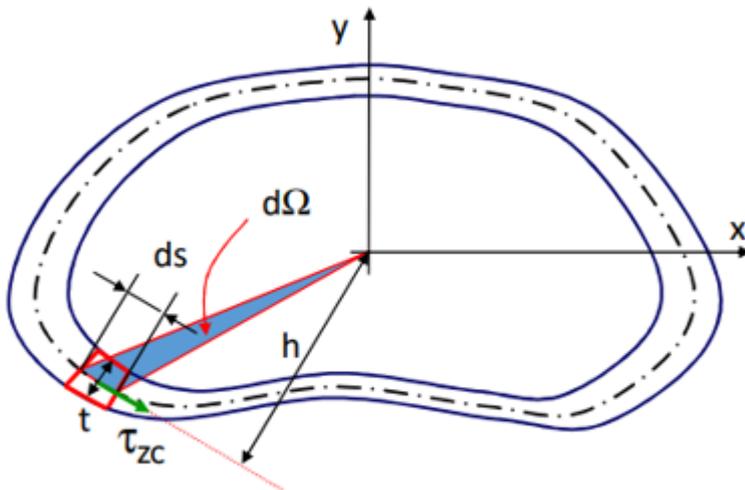
For hollow, thin-walled sections (like tubes or box beams), the shear stress flows around the perimeter. We assume the stress is constant across the thickness and tangent to the midline.

Key Concepts:

- **Shear Flow (q):** The product of stress and thickness is constant along the entire perimeter.

$$q = \tau_{zc} \cdot t = \text{constant}$$

- **Enclosed Area (Ω):** The area enclosed by the **midline** of the wall thickness (this is distinct from the material area).



Bredt's Formula (Stress):

Relates the torque to the shear stress and the enclosed area.

$$\tau_{zc} = \frac{M_z}{2\Omega t}$$

⚡ Critical Point (Design)

Since the shear flow q is constant, the **Maximum Shear Stress** occurs at the point where the wall thickness is **minimum**.

$$\tau_{max} = \frac{M_z}{2\Omega t_{min}}$$

Accuracy:

Bredt's formula is an approximation. As the wall gets thinner ($D_2/D_1 \rightarrow 1$), the result converges to the exact theoretical solution.

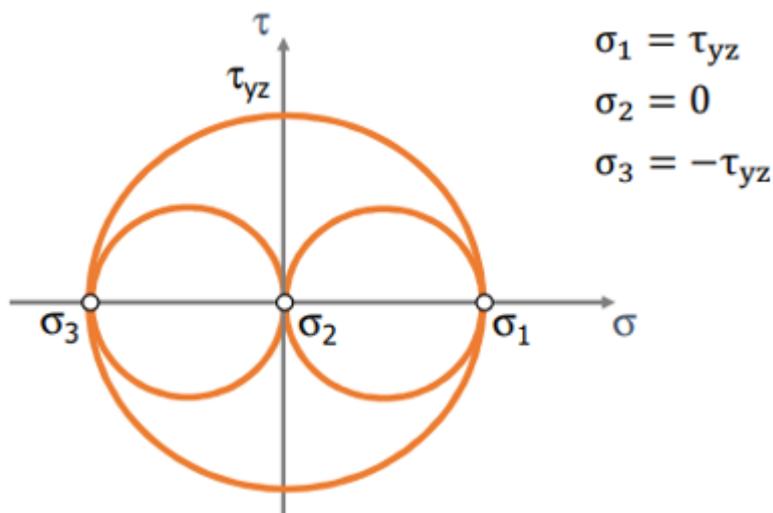
Stress State and Failure in Torsion

Torsion generates a state of **Pure Shear** on the cross-section.

A. The Stress Tensor & Mohr's Circle

The only non-zero components are shear stresses.

- **Center:** $(0, 0)$
- **Radius:** $R = \tau_{max}$



B. Principal Stresses

By rotating the element 45° , we find the principal stresses (max tension/compression):

$$\begin{aligned}\sigma_1 &= \tau_{max} \quad (\text{Tension}) \\ \sigma_2 &= 0 \\ \sigma_3 &= -\tau_{max} \quad (\text{Compression})\end{aligned}$$

C. Static **Failure** Criteria

How we calculate the "Ideal Stress" (σ_{id}) to check against the material's yield strength (σ_{yield}):

Material Type	Theory	Ideal Stress Formula	Failure Mode
Brittle	Max Normal Stress (Rankine)	$\sigma_{id} = \sigma_1 = \tau_{max}$	Fails in tension at 45° (helical fracture)
Ductile	Max Shear Stress (Tresca)	$\sigma_{id} = \sigma_1 - \sigma_3 = 2\tau_{max}$	Fails in shear on the cross-section plane

Shear Forces in Beams (Jourawsky)

When a beam carries a Shear Force (T_y or T_x), it generates a distribution of shear stresses τ across the section.

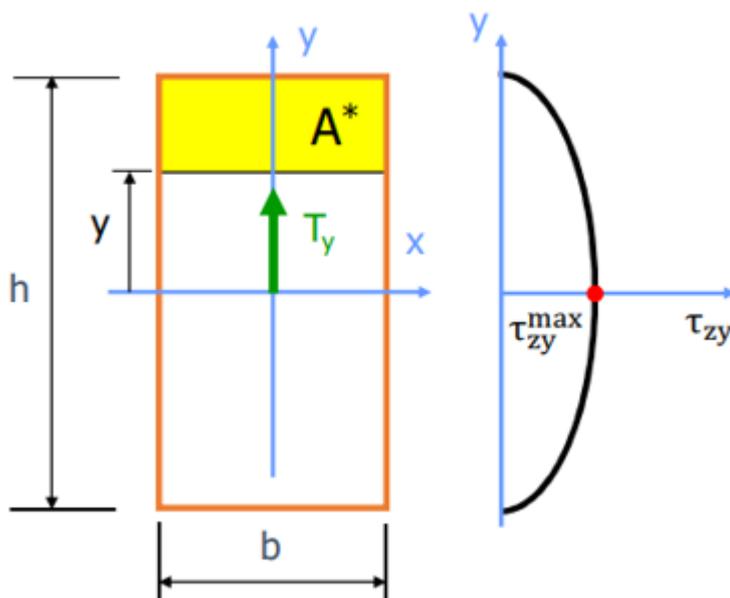
- **Origin:** Shear force is the derivative of the bending moment ($T_y = \frac{dM_x}{dz}$). Because M changes along z , the normal stresses σ_{zz} change, creating a "imbalance" that must be equilibrated by shear stress.

Jourawsky's Formula (The General Law)

Calculates the average shear stress across a chord of width b .

$$\tau_{zy} = \frac{T_y S_x^*}{b I_{xx}}$$

- T_y : Shear force acting on the section.
- I_{xx} : Second moment of area of the *entire* section.
- b : Width of the cross-section at the point of analysis (the "cut" width).
- S_x^* (The Tricky Part): The First Moment of Area of the *portion of the area above (or below) the cut*.



Common Cross-Sections (Reference)

A. Rectangular Section

The distribution is **Parabolic**.

- Maximum stress occurs at the Neutral Axis ($y = 0$).
- Formula:

$$\tau_{max} = \frac{3}{2} \frac{T_y}{A} = 1.5\tau_{avg}$$

B. Circular Section

The distribution is **Parabolic**.

- Maximum stress occurs at the Neutral Axis.
- Formula:

$$\tau_{max} = \frac{4}{3} \frac{T_y}{A} \approx 1.33\tau_{avg}$$

Comparison

In long, slender beams, the shear stress τ_{max} is usually negligible compared to the bending stress σ_{max} .

Thin-Walled Sections (C-Shapes, I-Beams)

For thin sections, the shear stress flows along the walls of the profile.

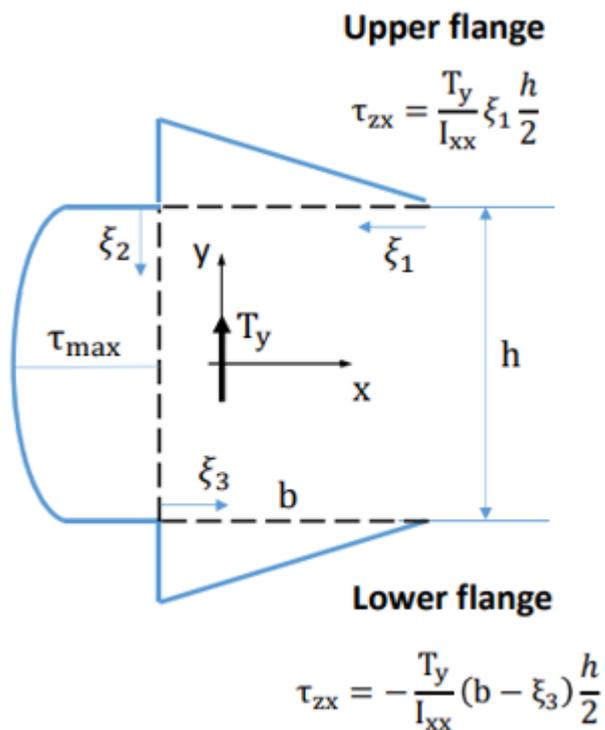
Analysis of a C-Section:

1. Flanges (Horizontal parts): Shear stress varies **Linearly**.

- It is zero at the free ends and maximum at the corner.
- Formula: $\tau_{zx} = \frac{T_y}{I_{xx}} s_1$ (where s_1 is distance from tip).

2. Web (Vertical part): Shear stress varies **Parabolically**.

- It starts from the value at the corner and peaks at the Neutral Axis.
- Formula: $\tau_{zy} = \frac{T_y}{s_2 I_{xx}} (\dots)$.



Shear Center

Summary

The Problem:

If a transverse shear force (T_y) is applied at the Centroid (G) of a cross-section that is *not* doubly symmetric (like a C-channel), it generates an unbalanced internal twisting moment. The beam will bend and twist.

The Solution:

The **Shear Center** (C or S) is the specific point in the cross-section plane where the shear force must be applied to produce **pure bending** (zero twist).

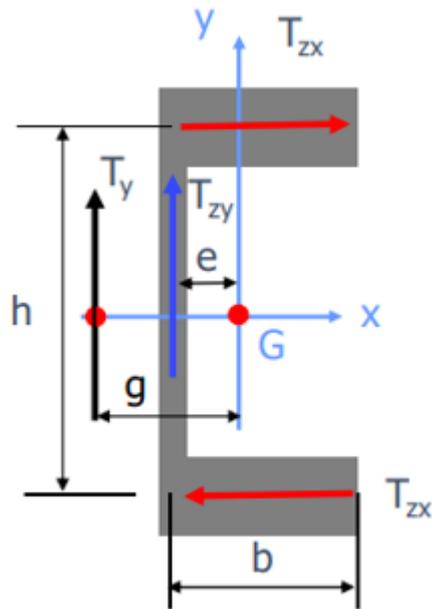
Analysis of a C-Section

1. **Shear Flow Imbalance:** The shear force T_y generates shear stresses in the web (T_{zy}) and horizontal stresses in the flanges (T_{zx}).
2. **Internal Torque:** The forces in the flanges (T_{zx}) are equal and opposite, creating a couple (torque) that tries to twist the section.

Determining the Position (g)

To find the location of the shear center (distance g from the centroid G), we enforce equilibrium: the moment generated by the external force must equal the moment

generated by the internal shear flow.



Moment Balance Equation (about G):

$$T_y \cdot g = T_{zx} \cdot h + T_{zy} \cdot e$$

$$g = \frac{T_{zx} \cdot h}{T_y} + e$$

- T_{zx} : Resultant shear force in the flange.
- h : Distance between flange centerlines.
- e : Distance from web centerline to Centroid G .

⚠ Effect of Misalignment

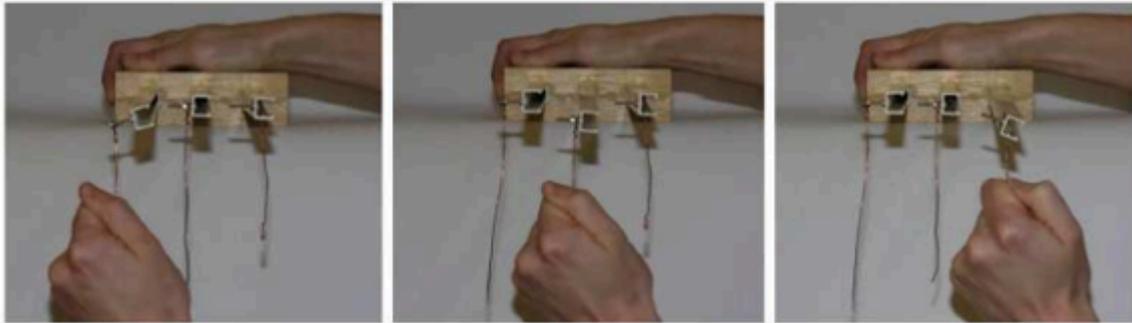
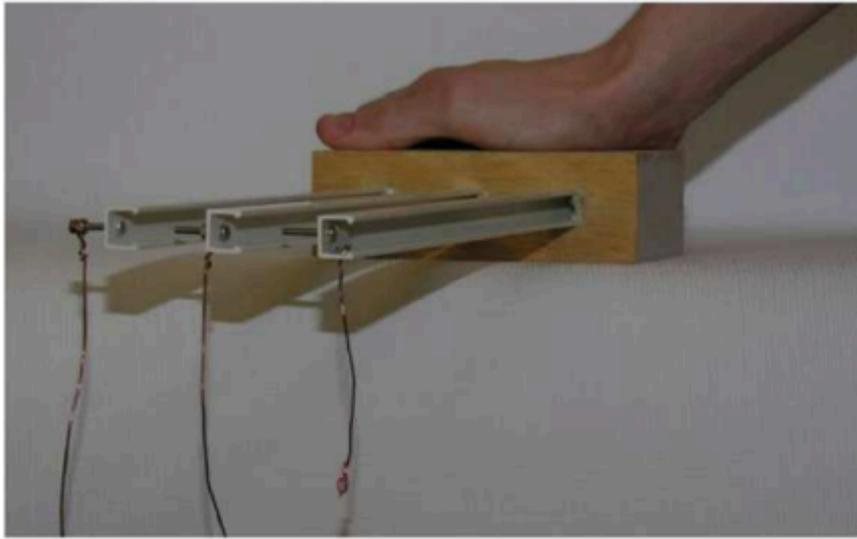
If the external load is applied at the Centroid G (and not the Shear Center), the system is statically equivalent to a Shear Force plus a **Twisting Moment**.

$$M_z = T_y \cdot g$$

This causes combined bending and torsion failures.

Visual Proof

The image below demonstrates the difference. The beams loaded at the centroid (left) twist significantly. The beam loaded at the shear center (right) bends straight.



[Source](#)

Ideal Stresses (Failure Criteria)

In a generic point of the beam, we simultaneously have Normal Stresses (σ_{zz}) and Shear Stresses (τ_{zx}, τ_{zy}).

To check for failure, we must combine these into an equivalent "Ideal Stress".

The Stress State

The total shear stress at a point is the vector sum of its components:

$$\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2}$$

The Principal Stresses ($\sigma_1, \sigma_2, \sigma_3$) for this state are:

$$\sigma_{1,3} = \frac{\sigma_{zz}}{2} \pm \sqrt{\left(\frac{\sigma_{zz}}{2}\right)^2 + \tau^2} ; \quad \sigma_2 = 0$$

Failure Hypotheses

Depending on the material (Brittle vs. Ductile), we use different formulas to calculate σ_{id} .

A. Rankine (Brittle Materials)

Based on the Maximum Normal Stress.

$$\sigma_{id} = \sigma_1 = \frac{|\sigma_{zz}|}{2} + \sqrt{\left(\frac{\sigma_{zz}}{2}\right)^2 + \tau^2}$$

B. Tresca (Ductile Materials)

Based on the Maximum Shear Stress. Conservative and safe.

$$\sigma_{id} = \sqrt{\sigma_{zz}^2 + 4\tau^2}$$

C. Von Mises (Ductile Materials - Most Common)

Based on Distortion Energy. This is the standard for metals like steel.

$$\sigma_{id} = \sqrt{\sigma_{zz}^2 + 3\tau^2}$$

The "Safety Check"

Once you calculate σ_{id} , the safety condition is simply:

$$\sigma_{id} \leq \sigma_{allowable} = \frac{\sigma_{yield}}{\text{Safety Factor}}$$

Buckling of Slender Beams (Euler Instability)

The Phenomenon:

Slender beams subjected to an axial compressive load (P) do not fail by crushing; they fail by sudden bending (buckling) sideways.

The Governing Equation

Unlike standard beam theory, here the internal bending moment depends on the deformation itself (v).

- Equilibrium: $M_x(z) = P \cdot v(z)$.
- Differential Equation:

$$\frac{d^2v}{dz^2} + \frac{P}{EI_{min}}v = 0$$

Euler's Critical Load (P_{cr})

The load at which the beam becomes unstable and buckles. It always happens about the **weakest axis** (I_{min}).

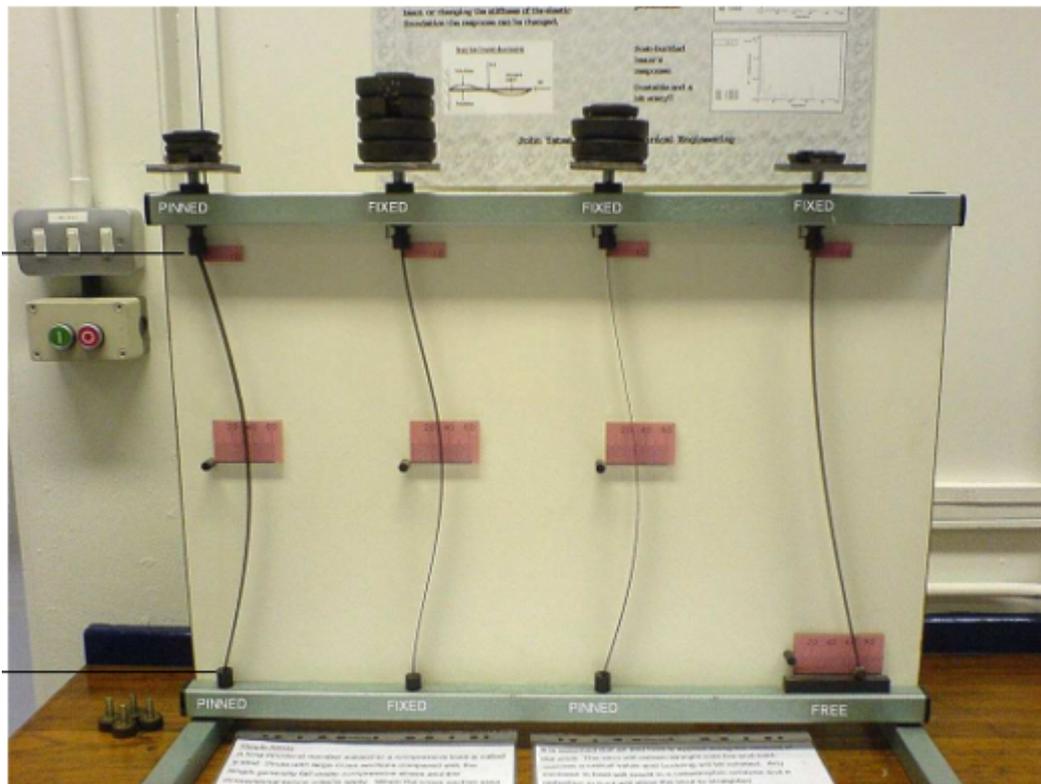
$$P_{cr} = \pi^2 \frac{EI_{min}}{l_0^2}$$

- E : Young's Modulus.
- I_{min} : Minimum Moment of Inertia of the section.
- l_0 : Effective Length of the column.

Effect of Boundary Conditions (l_0)

The effective length l_0 depends on how the beam is supported.

Supports	Effective Length (l_0)	Critical Load Capacity
Pinned - Pinned	L	$1 \times P_{Euler}$
Fixed - Free (Cantilever)	$2L$	$0.25 \times P_{Euler}$ (Weakest)
Fixed - Fixed	$0.5L$	$4 \times P_{Euler}$ (Strongest)
Fixed - Pinned	$0.7L$	$2 \times P_{Euler}$



Critical Stress and Slenderness

We define the Slenderness Ratio (λ) to determine if a beam will yield or buckle.

$$\lambda = \frac{l_0}{\rho_{min}} \quad \text{where} \quad \rho_{min} = \sqrt{\frac{I_{min}}{A}} \quad (\text{Radius of Gyration})$$

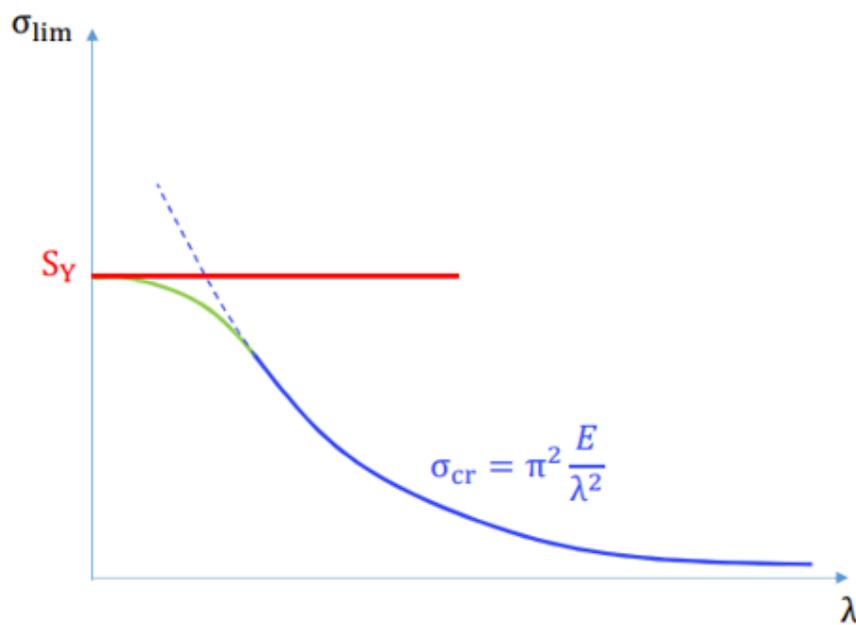
Critical Stress:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \pi^2 \frac{E}{\lambda^2}$$

⚡ Failure Modes

- Short / Squat Beams (Low λ): Fail by Yielding ($\sigma_{cr} = \sigma_{yield}$).
- Long / Slender Beams (High λ): Fail by Buckling ($\sigma_{cr} < \sigma_{yield}$).

You must check which curve controls the design.



$$F_S = \frac{S_Y}{\sigma_{id}}$$

$$F_S = \frac{\sigma_{cr}}{|\sigma|}$$

8. Stress Concentration

8. Stress Concentration

The Phenomenon:

The formulas derived from [Saint-Venant](#) (e.g., $\sigma = N/A$) assume the cross-section changes gradually or is constant.

In reality, machine parts have abrupt changes like holes, grooves, or shoulders (called **Notches** or **Stress Raisers**). These features cause the stress lines to "crowd" together, creating a local peak stress (σ_{max}) significantly higher than the average.

The Theoretical Stress Concentration Factor (K_t)

We quantify this effect using a simple multiplier, the **Geometric Stress Concentration Factor** (K_t).

$$K_t = \frac{\sigma_{max}}{\sigma_{nom}}$$

- σ_{max} : The actual peak stress at the root of the notch (principal stress).
- σ_{nom} : The "Nominal Stress" calculated using standard formulas (N/A , My/I) on the **Net (Minimum) Cross-Section**.

Warning: The Net Area

When calculating σ_{nom} , you must **always** use the dimensions of the **reduced section** (diameter d), not the gross section (diameter D).

Methods of Calculation

Since standard beam theory fails at the notch, how do we find K_t ? There are three main approaches:

A. Analytical Methods (Historical)

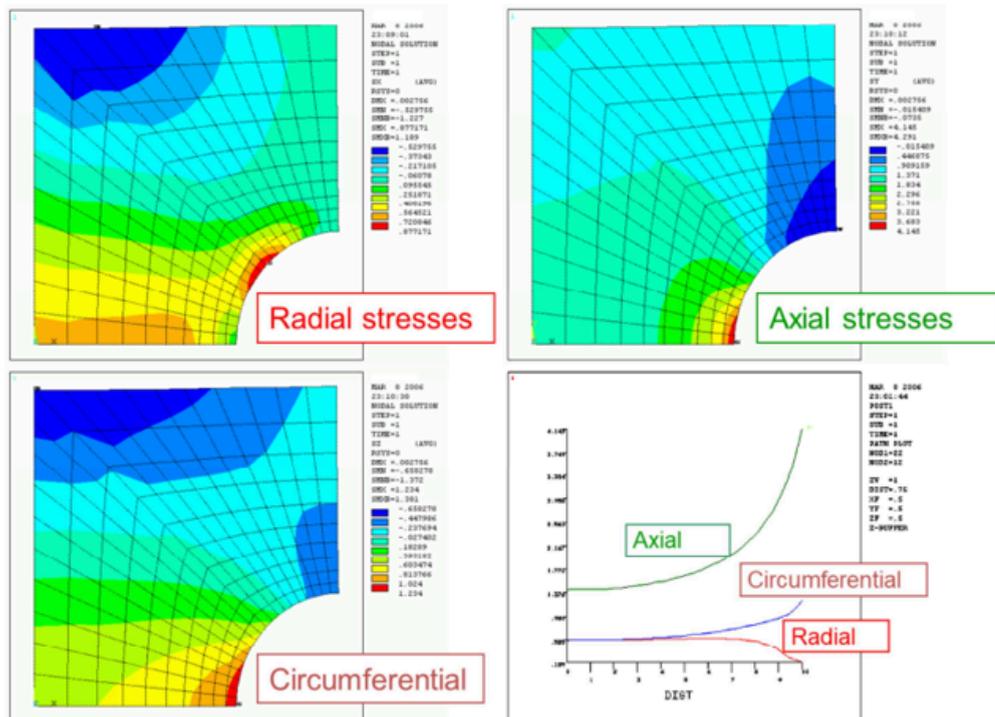
- **Kirsch (1898)**: Solved the stress state around a circular hole in an infinite plate.
- **Neuber (1930)**: Contributed significant solutions for various geometries.
- *Limitation*: Exact analytical solutions only exist for very simple shapes.

B. Numerical Methods (Modern Standard)

Using computer simulations to solve the elasticity equations approximately.

- **FEM (Finite Element Method):** Discretizes the part into a mesh.
- **BEM (Boundary Element Method):** Discretizes only the surface.
- *Visual Evidence:* The FEM solution for a grooved bar shows how axial stress peaks at the root of the notch while radial stress develops inside.

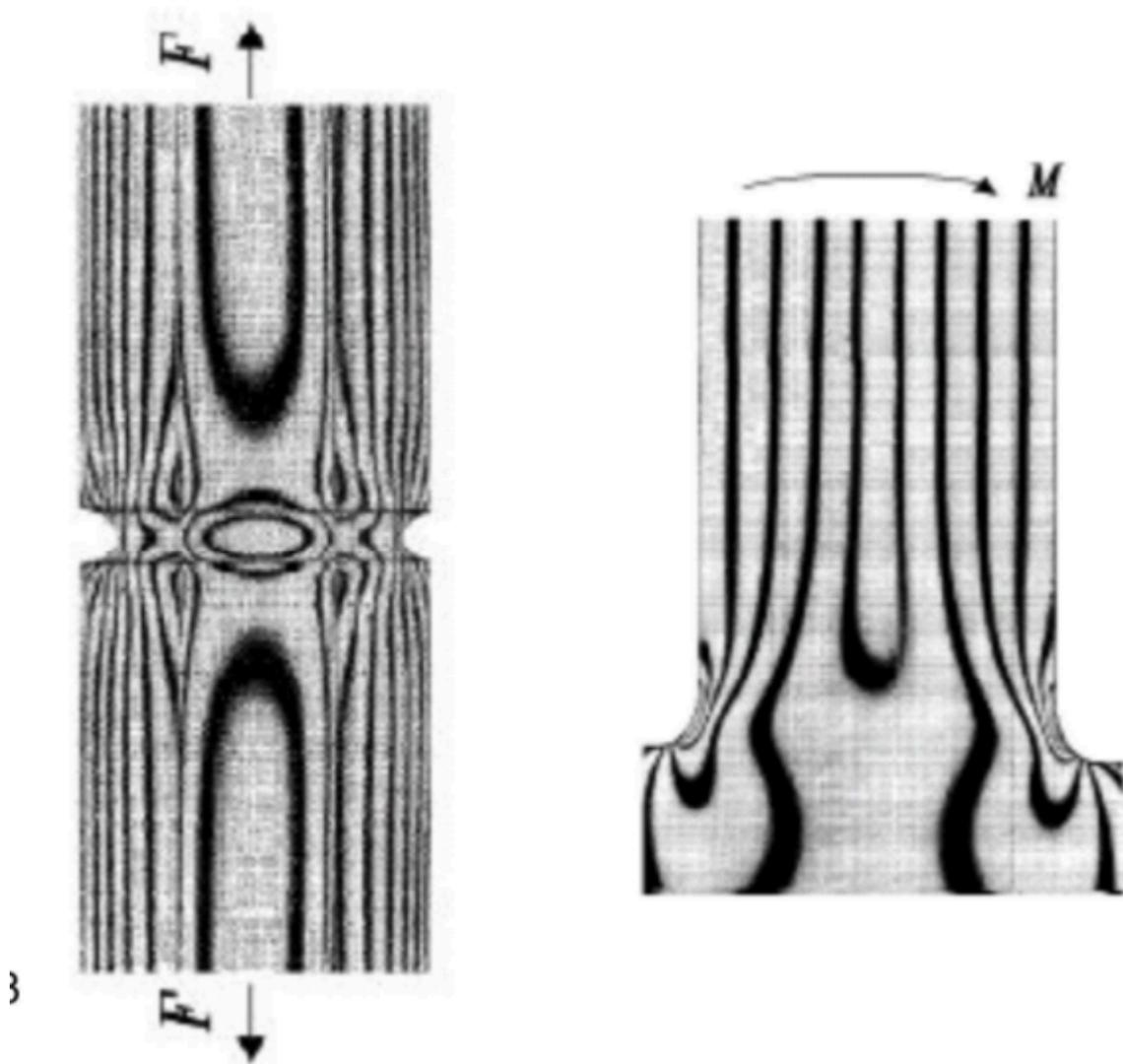
Stress concentration FEM solution: grooved round bar (in tension)



C. Experimental Methods

Used to validate theories or test complex parts.

- **Photoelasticity:** Uses optical properties of stressed plastic models to visualize stress fringes.
- **Strain Gauges:** Measures strain at specific points.
- **Brittle Coatings:** Cracks form in the coating perpendicular to max tensile stress.



3. Dependencies

The value of K_t depends only on:

1. **Geometry:** The sharpness of the notch (radius r) and the size of the transition (D/d).
2. **Loading Mode:** A specific notch behaves differently under Tension, Bending, or Torsion.

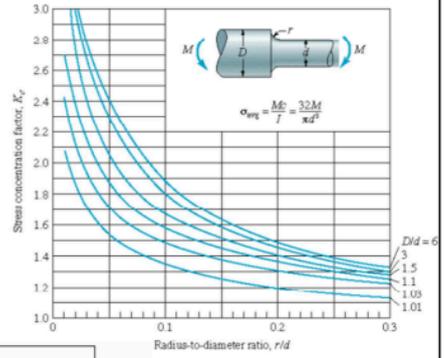
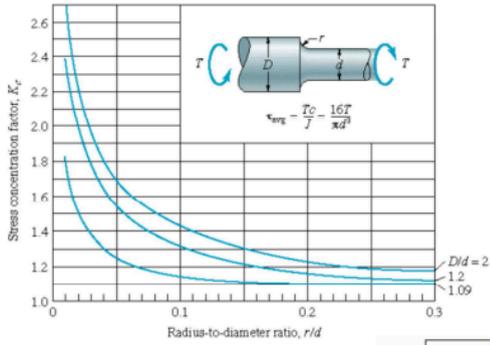
4. Peterson's Charts (Design Tool)

Engineers use published charts (from R.E. Peterson) to find K_t quickly without running a simulation.

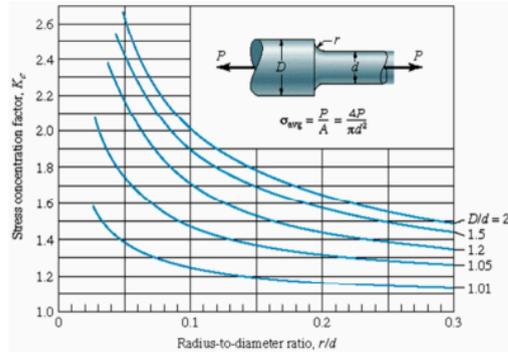
Procedure:

1. Calculate geometric ratios: r/d (x-axis) and D/d (curve selection).
2. Read K_t from the y-axis.
3. Calculate peak stress: $\sigma_{max} = K_t \cdot \sigma_{nom}$.

Stress concentration factors Round shaft with shoulder fillet



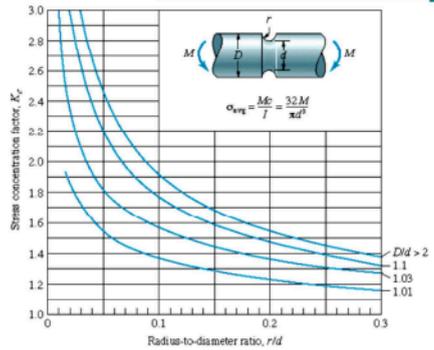
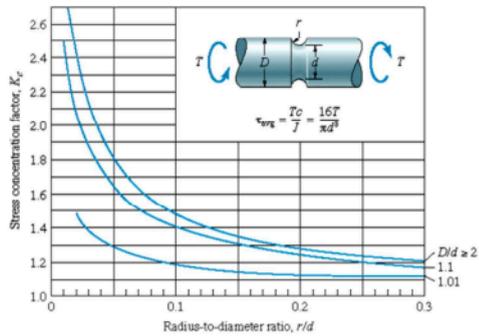
Tension



Torsion

Bending

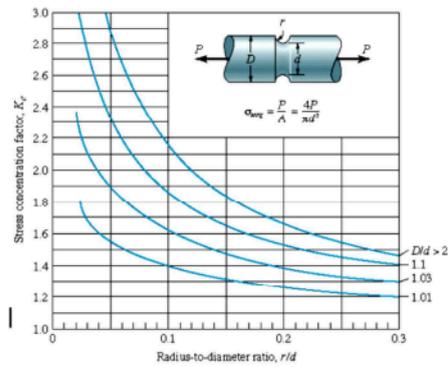
Stress concentration factors Grooved round bar



Tension

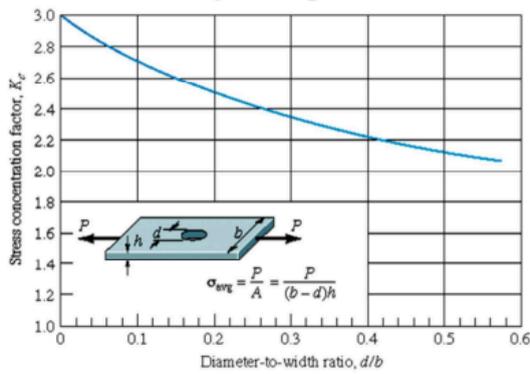
Torsion

Bending

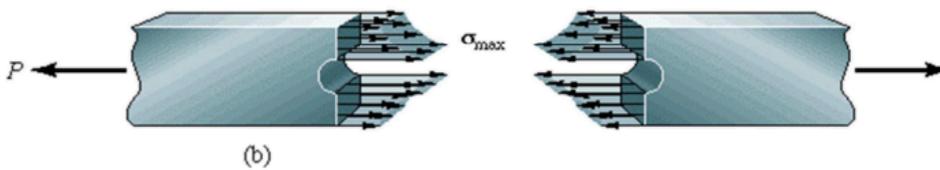
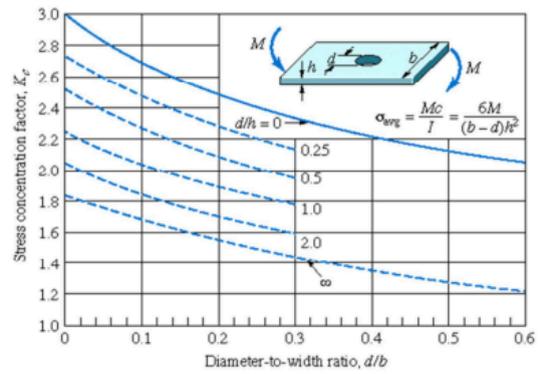


Stress concentration factors Bar (plate) with a transverse hole

Tension/compression

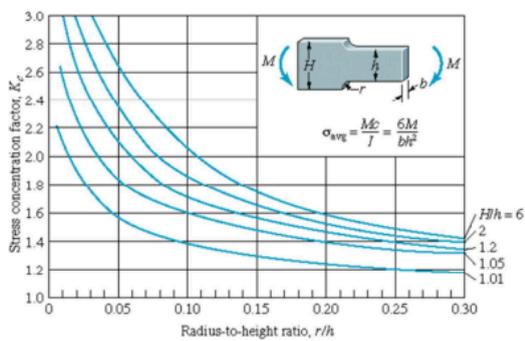
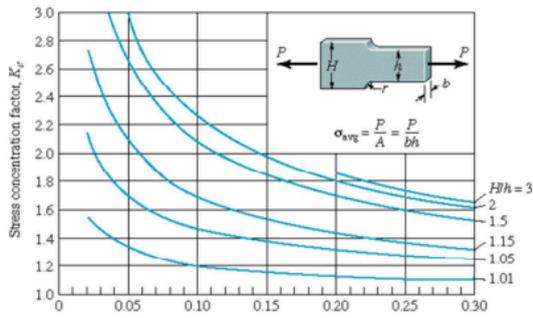


Bending



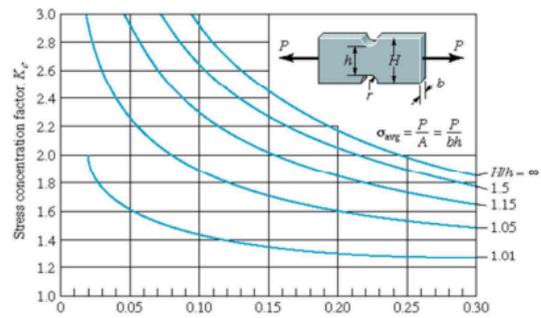
Stress concentration factors Rectangular bars (plates)

Rectangular filleted bar

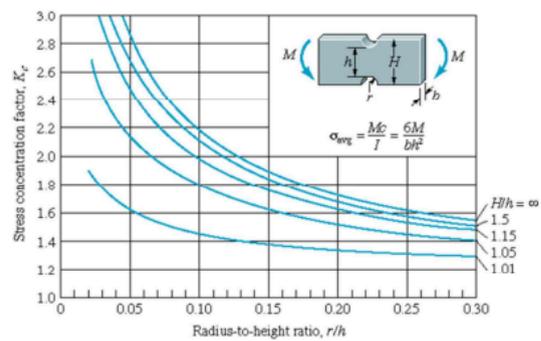


Notched rectangular bar

Tension/compression



Bending

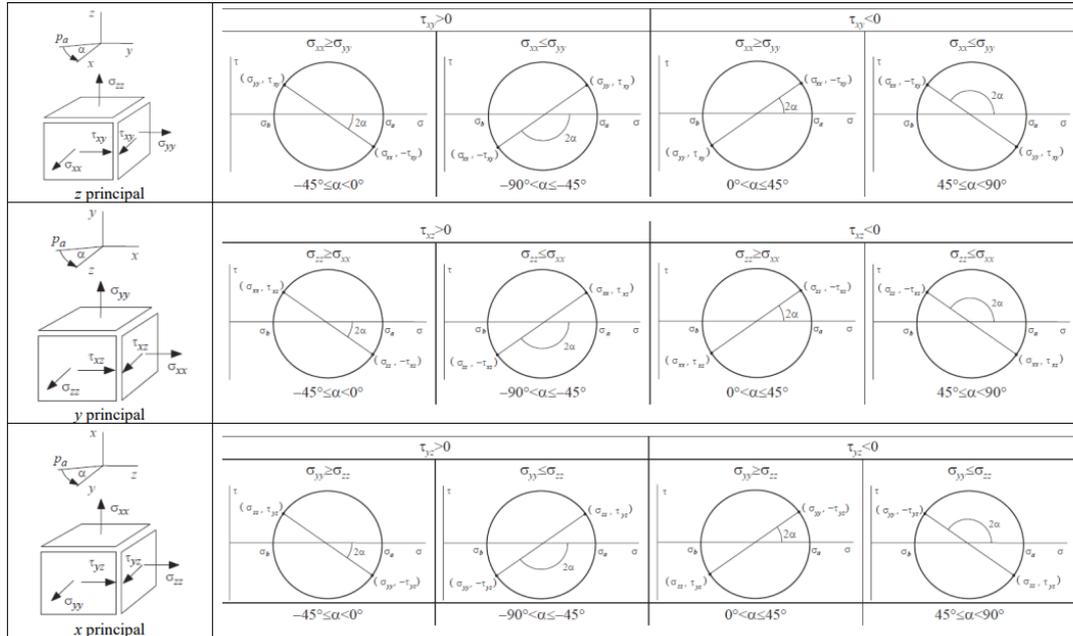


Mohr's Circle

Mohr's Circle

Mohr's Circles (inverse construction)

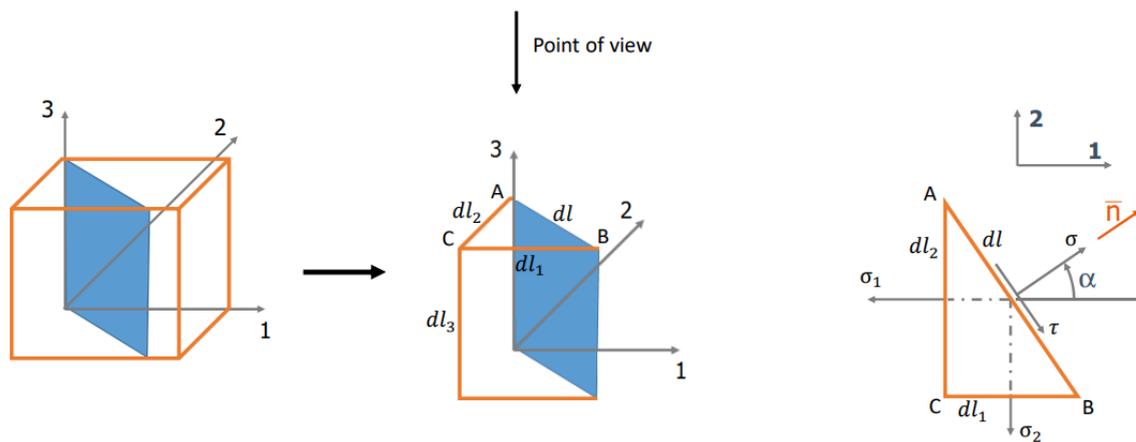
Angle α is the angle between the principal reference 1-2-3 and the generic reference system x - y - z (p_a denotes the principal axis corresponding to σ_a).



🔗 The Purpose

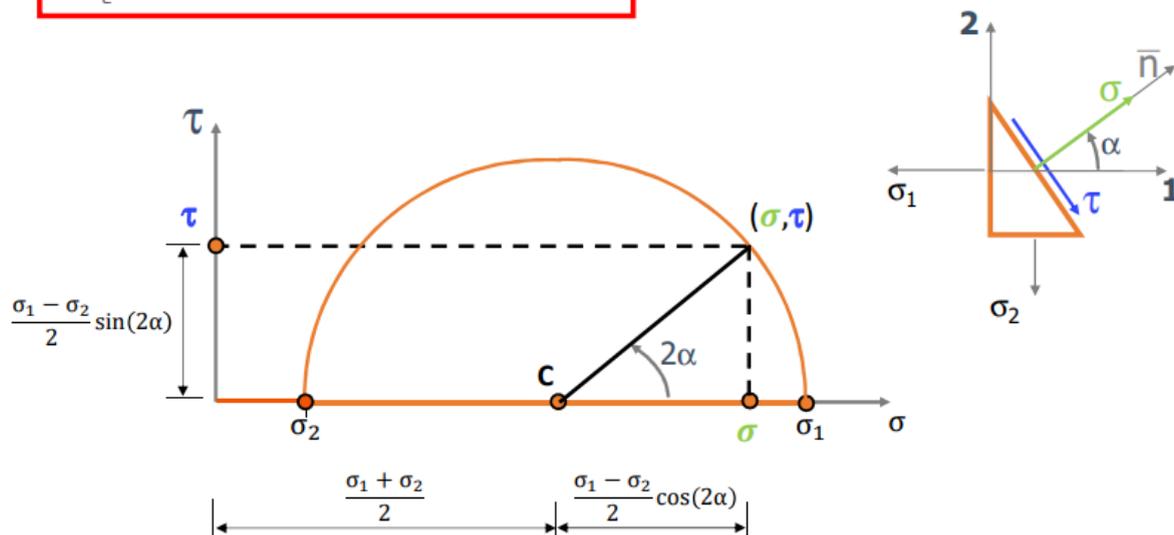
We know the stress state ($\sigma_x, \sigma_y, \tau_{xy}$) on the horizontal/vertical faces. Mohr's Circle allows us to find the stress on any inclined plane (θ) graphically, without solving complex rotation equations.

Most importantly, it reveals the **Principal Stresses** (Max/Min normal stress) and the **Maximum Shear Stress**.



$$\begin{cases} \sigma = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos(2\alpha) \\ \tau = \frac{\sigma_1 - \sigma_2}{2} \sin(2\alpha) \end{cases}$$

A circle with center $C = \left(\frac{\sigma_1 + \sigma_2}{2}, 0\right)$ and radius $r = \frac{\sigma_1 - \sigma_2}{2}$



The Sign Convention

- Normal Stress (σ):
 - Tension (+): Pulling away from the element (Right on the graph).
 - Compression (-): Pushing into the element (Left on the graph).
- Shear Stress (τ):
 - Standard Convention:
 - Face X: If τ rotates the element Counter-Clockwise \rightarrow Plot Down (-).
 - Face X: If τ rotates the element Clockwise \rightarrow Plot Up (+).
 - Note: The vertical axis is usually τ (positive down) or τ (positive up).

Construction Steps

1. Identify the Points:

- **Point X (Face X):** Plot $(\sigma_x, -\tau_{xy})$
- **Point Y (Face Y):** Plot (σ_y, τ_{xy})
- *Note:* One point will be above the axis, one below.

2. Find the Centre (C):

The circle lies on the σ axis (horizontal).

$$C = \frac{\sigma_x + \sigma_y}{2}$$

- This is the **Average Normal Stress** (σ_{avg}).

3. Calculate the Radius (R):

Distance from Centre C to Point X.

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Critical Values

Once drawn, the circle reveals the failure points immediately:

1. Principal Stresses (σ_1, σ_2):

The points where the circle crosses the horizontal σ -axis. Shear is **Zero** here.

- **Max Tension/Compression:**

$$\sigma_{1,2} = C \pm R$$

2. Maximum Shear Stress (τ_{max}):

The highest/lowest point of the circle (top/bottom).

$$\tau_{max} = R$$

3. Principal Orientation (θ_p):

The angle on the circle (2θ) is **double** the physical angle (θ).

- If you rotate 90° on the circle (from Point X to σ_1), you rotate 45° on the real element.

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Special Cases

- **Uniaxial Tension:** Circle touches the origin. $\sigma_2 = 0$.
- **Pure Shear:** Centre is at origin ($C = 0$). $\sigma_1 = -\sigma_2$.

- Hydrostatic Pressure: $\sigma_x = \sigma_y$. Radius = 0. The circle is a Dot.

Mohr's Circle (Inverse Construction)

🔗 The Goal

You start with the **Principal Stresses** (σ_1, σ_2) and want to find the normal (σ) and shear (τ) stress on a plane inclined at a specific angle θ .

The Setup

1. Plot Principal Stresses

- Mark σ_1 (Max) and σ_2 (Min) on the horizontal σ -axis.
- *Note:* Shear is zero at these points.

2. Draw the Circle

- **Centre (C):** Midpoint between σ_1 and σ_2 .
- **Radius (R):** Distance from C to σ_1 .
- Draw the full circle.

The Rotation Rule (2θ)

⚡ The Golden Rule of Mohr

Real World vs. Circle World

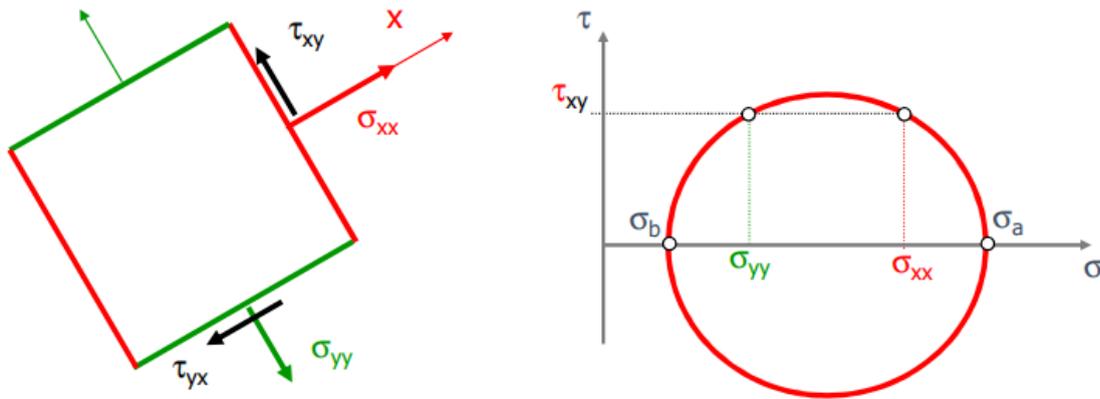
- **Real Element:** Angle is θ .
- **Mohr's Circle:** Angle is 2θ .
- **Direction:** The direction is the **SAME**.
 - If you rotate the cut **Counter-Clockwise (CCW)** by θ in reality, you rotate the radius **Counter-Clockwise (CCW)** by 2θ on the circle.

The Procedure

1. **Start at the Reference Point:** Usually Point 1 ($\sigma_1, 0$), which represents the Principal Plane (where max stress acts).
2. **Rotate:** Measure an angle of 2θ from the horizontal axis in the correct direction.
3. **Find Point P:** The end of this new radius is your state ($\sigma_\theta, \tau_\theta$).

4. **Read Values:** The coordinates of P are the Normal and Shear stress on that inclined plane.

The “Pole” Method (Origin of Planes)



This is a graphical trick to find the stress on any plane without calculating angles first. It is often faster for complex problems.

1. Find the Pole (P)

- Start at a known point on the circle (e.g., the Principal Stress point $\sigma_1, 0$).
- Draw a line through that point **PARALLEL** to the physical face it represents (e.g., since σ_1 acts on a vertical plane, draw a vertical line through the point).
- Where this line crosses the circle again is the **Pole (P)**.

2. Use the Pole

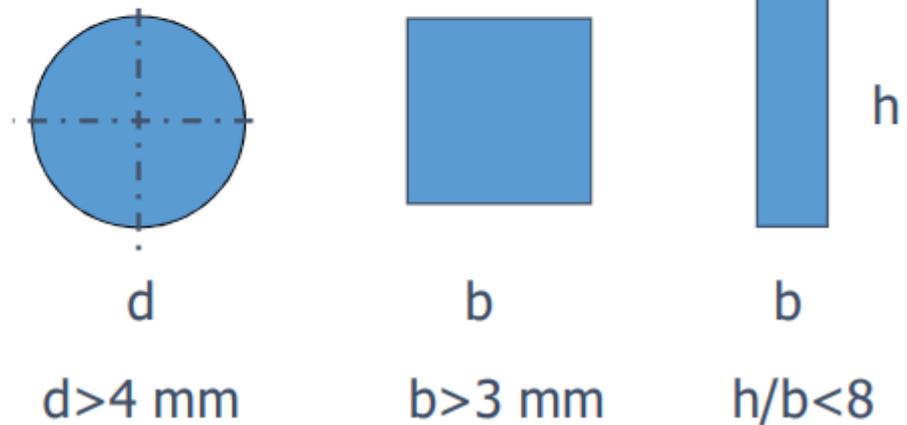
- Draw a line from the Pole (P) parallel to **any physical cut** you want to analyse (angle θ).
- The point where this line hits the circle gives you the exact (σ, τ) for that cut.

Tensile Test

Tensile Test

The Specimen

- According to the European standard EN 10002/1
 - Testing temperature: $23 \pm 5^\circ\text{C}$
 - Types of cross-sections:



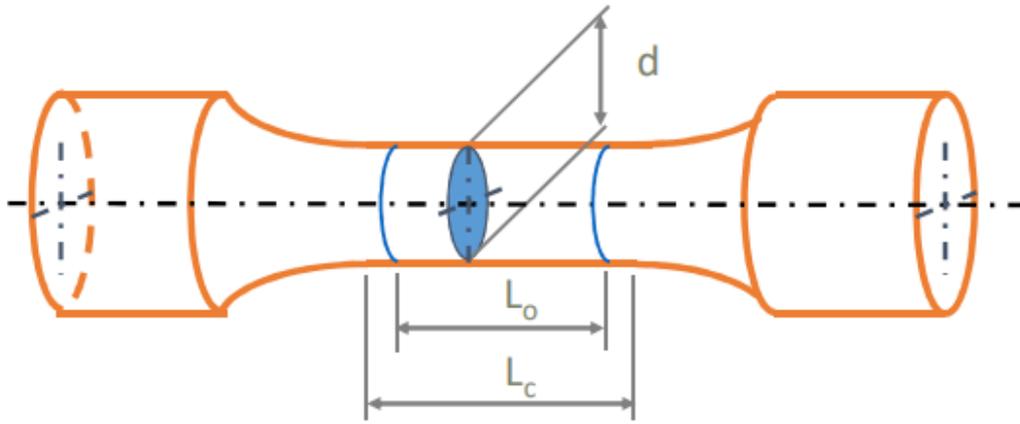
Standardization



Specimens are standardized to allow for consistent results across labs.

Specimens are said to be **proportional** when gauge length L_0 is proportional to the square root of the area of the cross-section A_0 .

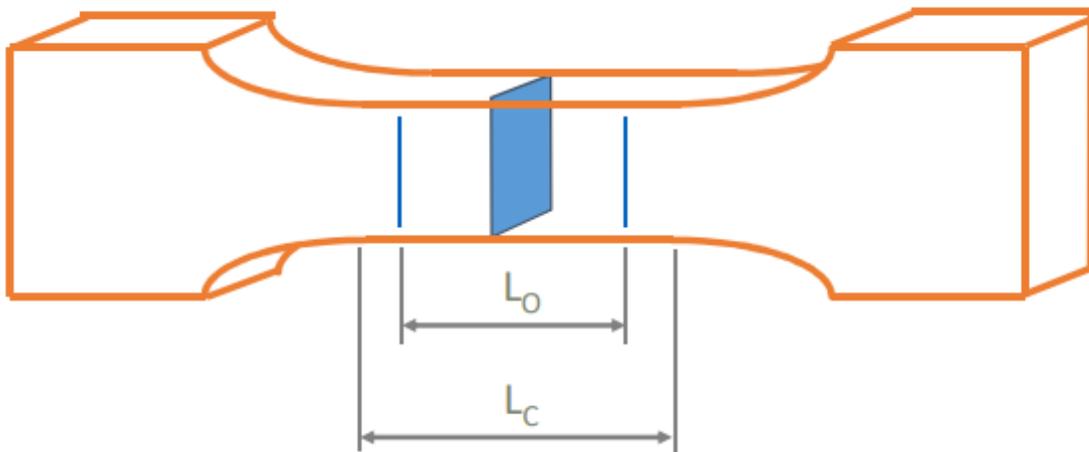
Circular Cross-Section



$$L_0 = 5d \text{ Rounded to nearest 5mm}$$

$$L_0 + \frac{d}{2} < L_C < L_0 + 2d$$

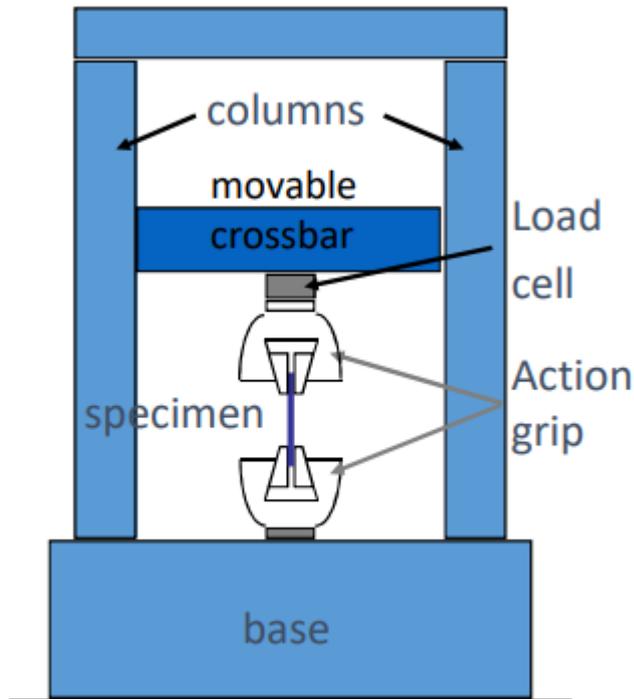
Rectangular Cross-Section



$$L_0 = 5.65\sqrt{A_0} \text{ Rounded to nearest 5mm}$$

$$L_0 + 1.5\sqrt{A_0} \leq L_C \leq L_0 + 2.5\sqrt{A_0}$$

The Setup



The specimen is slowly pulled, measuring the elongation $\Delta L = L - L_0$.

⚡ Loading rate matters!

We therefore have the following standard rates for steel and aluminium respectively (units in $Nmm^{-2}s^{-1}$):

$$6 \leq \frac{d\sigma}{dt} \leq 30$$

$$2 \leq \frac{d\sigma}{dt} \leq 10$$

Note that in the test, we have stress in one direction, and ideally, this should be the maximum [principal stress](#) σ_0 , therefore giving the stress tensor:

$$[\sigma] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we want to test out maximum tangential stress, we apply the tension at a 45° angle with slip planes, leading to:

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}$$

This is the maximum shear stress given by the fractured plane. This formula can be visualized using [Mohr's Circle](#): the highest point on the graph is the top of the circle spanning from the minimum stress to the maximum stress.

Results

From any point we can get [strain](#) percent:

$$\varepsilon = \frac{\Delta L}{L_0} = \frac{L_i - L_0}{L_0}$$

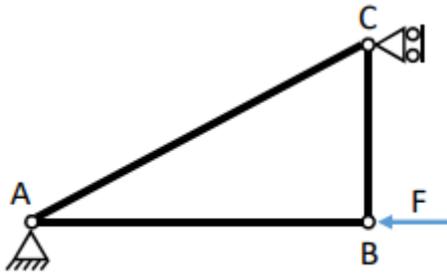
We can also get [poisson's ratio](#) by using:

$$\varepsilon_x = \varepsilon_y = -\nu\varepsilon_z$$

For isotropic materials.

Trusses

Trusses



✎ External forces are only applied to the internal hinges (nodes).

In each rod, the only internal action is the normal load N

- If $N > 0$, then the beam is loaded by traction
- If $N < 0$, then the beam is loaded by compression

✓ Simplified [degree of static determinacy](#) ✓

$h = \text{Number of Rods} + \text{Number of External Reactions} - 2 \cdot \text{Number of Internal Hinges}$

Solution Strategy

1. Compute h (Check if isostatic).
2. **External Reactions:** Treat the whole truss as a *single rigid body* first. Ignore the insides. Solve for the ground supports ($H_A, V_A, \text{etc.}$).
3. **Internal Forces (N):**
 - **Method of Nodes:** Solves *every* rod (good for full analysis).
 - **Method of Sections (Ritter):** Solves *specific* rods (good for finding just one or two forces).

Method 1: Equilibrium at Nodes

The Logic

Isolate a specific node (hinge) and treat it as a particle. Since the node is in equilibrium, the sum of forces acting on it must be zero.

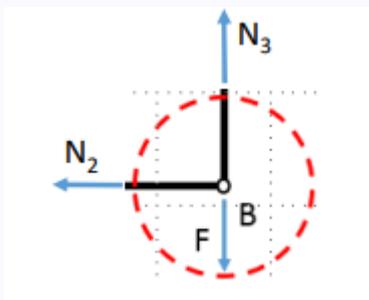
Procedure:

1. **Isolate a Node:** Start with a node where you have **at least 1 known force** and **no more than 2 unknown rods** (since you only have 2 equations: $\sum F_x, \sum F_y$).
2. **Draw Vectors:**
 - Draw known external forces.
 - Draw unknown rod forces (N) pointing **AWAY** from the node.
 - *Note:* We always assume **Traction (+)** initially.
3. **Solve:**
 - $\sum F_x = 0$
 - $\sum F_y = 0$
4. **Interpret signs:**
 - Result > 0 : Assumption correct (Traction).
 - Result < 0 : Rod is in Compression.
5. **Repeat:** Move to the next connected node and use the values you just found.

Spotting "Zero Force" Members

Sometimes you can spot rods with $N = 0$ just by looking.

- *Example:* Look at Node B. A vertical force F balances the vertical rod (N_3). However, **nothing** opposes the horizontal rod (N_2). Therefore, to maintain equilibrium, N_2 must be zero.



Method 2: Method of Sections (Ritter)

The Logic

Instead of solving node-by-node, you "cut" the truss into two separate rigid bodies. The internal forces of the cut rods become external forces on the new sections.

The Rules of the Cut:

1. **Cut Max 3 Rods:** You only have 3 equilibrium equations ($\sum F_x, \sum F_y, \sum M$), so you cannot solve for more than 3 unknowns.
2. **No Common Node:** The three cut rods cannot all meet at the same node (otherwise the moment equation becomes useless).

The "Ritter" Strategy:

To find the force in a specific rod (e.g., Rod A) without doing extra maths:

1. **Identify the other two rods** you cut (Rods B and C).
2. **Find their intersection point** (Point P).
3. **Sum Moments about Point P** ($\sum M_P = 0$).
 - Since Rods B and C pass through P, their moment is **zero**.
 - You are left with an equation containing *only* Rod A.